THE SEPTEMBER MEETING OF THE SAN FRANCISCO SECTION.

THE tenth regular meeting of the San Francisco Section of the AMERICAN MATHEMATICAL SOCIETY was held at the University of California on Saturday, September 29, 1906. The following sixteen members were present:

Professor R. E. Allardice, Professor H. F. Blichfeldt, Mr. A. J. Champreux, Professor G. C. Edwards, Professor M. W. Haskell, Professor L. M. Hoskins, Professor D. N. Lehmer, Professor A. O. Leuschner, Dr. J. H. McDonald, Dr. W. A. Manning, Professor H. C. Moreno, Professor C. A. Noble, Dr. T. M. Putnam, Professor Irving Stringham, Mr. L. C. Walker, Professor E. J. Wilczynski.

Professor R. E. Allardice occupied the chair at both the morning and afternoon sessions. The following officers were elected for the ensuing year: Professor E. J. Wilczynski, chairman; Dr. W. A. Manning, secretary; Professor M. W. Haskell, Professor H. F. Blichfeldt, and Dr. W. A. Manning, program committee. It was arranged to hold the next meeting on Saturday, February 23, at Stanford University.

The following papers were read at this meeting:

(1) PROFESSOR R. E. ALLARDICE: "Additional note on the multiple points of unicursal curves."

(2) PROFESSOR H. F. BLICHFELDT: "A theorem concerning the Sylow subgroups of simple groups."

(3) PROFESSOR M. W. HASKELL: "On the collineation group belonging to triangles quadruply in perspective."

(4) PROFESSOR L. M. HOSKINS: "The effect of viscosity on the propagation of free vibrations in an elastic solid."

(5) DR. W. A. MANNING: "On the order of primitive groups."

(6) PROFESSOR R. E. MORITZ: "On the symbolic representation of quotiential coefficients of the second order."

The paper by Professor Moritz was read by title. Abstracts of the papers follow below. The abstracts are numbered to correspond to the titles in the list above.

1. If a unicursal curve be given by the equations $x = (a_0, a_1, \cdots, a_n)(\lambda, 1)^n$, etc., the multiple points depend on the

quantities $A = (a_0, b_1, c_2)$, etc. It is known that the multiple point of a cubic curve is given by the hessian of the cubic $(A, B, C, D)(\lambda, 1)^3$. In the case of curves of a higher order the problem becomes complicated on account of the fact that the quantities (a_0, b_1, c_2), (a_0, b_1, c_2), etc., are not independent. In a previous paper Professor Allardice gave a theorem for the quartic curve analogous to the theorem for the cubic stated above; and in an additional note, read at this meeting, curves of the fifth degree in space of two, three and four dimensions were considered.

2. Frobenius has proved ("Ueber auflösbare Gruppen V," *Sitzungsberichte der Berliner Akademie*, 1901, page 1324) that in a simple group G of composite order there must be contained a subgroup H whose order is a power of a prime p, and a substitution S of order prime to p which transforms H into itself yet is not commutative with every substitution of H. Professor Blichfeldt proves that we may use for H a subgroup of G of order p^λ, this being the highest power of p which divides the order of G, provided such a subgroup is abelian, or of some other specified type.

3. Professor Haskell showed that it is possible to construct a system of four triangles, each pair quadruply in perspective and inscribed in a conic, and that the perspective reflections thus defined generate a group of 72 collineations which is equivalent to the group

$$y_1 : y_2 : y_3 = \gamma^\lambda x_p : \gamma^\mu x_q : \gamma^\nu x_r,$$

where $\gamma^6 = 1$, $\lambda + \mu + \nu \equiv 0 \pmod 6$ and p, q, r are 1, 2, 3 in any order. This group might well be regarded as trivial, apart from its relation to the above configuration. This configuration exhibits further an interesting relation to the nine points of inflexion of a cubic and their harmonic polars.

4. In an isotropic elastic solid without internal friction or viscosity, dilatational and rotational vibrations travel with velocities which depend upon the elastic moduli and the density but are independent of the frequency. The object of Professor Hoskins's paper is to investigate how this result is affected by viscosity. The discussion is restricted to the case in which viscosity is specified by a single modulus ν, the differential

equations for the displacements being obtained from those applying to a non-viscous solid by substituting $n + \nu d/dt$ for the rigidity-modulus n. It is found that the velocity of propagation is not independent of the frequency, and that for each kind of vibration it may have any value from 0 up to the value corresponding to no viscosity. It thus appears that viscosity furnishes a sufficient explanation of the fact that earthquake vibrations lasting but a fraction of a minute near their place of origin are recorded for several hours at distant places; the explanation being that the original complex disturbance analyzes into simple vibrations which travel with velocities varying from the greatest value (corresponding to no viscosity) down to zero.

5. Dr. Manning presented the following theorem: Let q be one of the numbers 2, 3, 4; p any prime greater than $q+1$; the degree of a primitive group which contains a substitution of order p with q cycles (without containing the alternating group) cannot exceed $pq+q$. This is directly connected with the similar theorem stated, in part without proof, by Jordan in the *Bulletin de la Société mathématique de France*, volume 1, page 221.

6. Professor Moritz's paper deals with analogies between certain differentiation formulas and the corresponding quotientiation formulas. It is first shown that for a function of n dependent variables $y = F(u_1, u_2, \cdots, u_n)$

$$\frac{qy}{qx} = \left(\frac{qy}{qu_1}\right)\frac{qu_1}{qx} + \left(\frac{qy}{qu_2}\right)\frac{qu_2}{qx} + \cdots + \left(\frac{qy}{qu_n}\right)\frac{qu_n}{qx},$$

where the parentheses are used in the eulerian sense to denote partial variations of the variables. The second quotiential coefficient is given by

$$\frac{q^2y}{qx^2} \quad \frac{qy}{qx} = \sum_{i=1}^{n}\left(\frac{qy}{qu_i}\right)\left(\frac{q^2y}{qu_i^2}\right)\overline{\frac{qu_i}{qx}}^2 + 2\sum_{\substack{k=2 \\ i<k}}^{k=n}\left(\frac{qy}{qu_k}\right)\left(\frac{q^2y}{qu_iqu_k}\right)\frac{qu_i}{qx}\frac{qu_k}{qx} + \sum_{i=1}^{n}\left(\frac{qy}{qu_i}\right)\frac{qu_i}{qx}\frac{q^2u_i}{qx^2}.$$

It is pointed out that this expression may be written symbolically thus

$$\frac{q^2y}{qx^2}\frac{qy}{qx}=\left[\sum_{i=1}^{n}\left(\frac{qy}{qu_i}\right)\frac{qu_i}{qx}\right]^{(2)}+\sum_{i=1}^{n}\left(\frac{qy}{qu_i}\right)\frac{qu_i}{qx}\frac{q^2u_i}{qx^2},$$

where the exponent in parenthesis signifies that the expression to which it is attached is to be squared, and after squaring

$$\left(\frac{qy}{qu}\right)^2,\quad \left(\frac{qy}{qu}\right)\left(\frac{qy}{qv}\right),\quad \left(\frac{qy}{qv}\right)^2$$

are to be replaced by

$$\left(\frac{qy}{qu}\right)\left(\frac{q^2y}{qu^2}\right),\quad \left(\frac{qy}{qv}\right)\left(\frac{q^2y}{quqv}\right),\quad \left(\frac{qy}{qv}\right)\left(\frac{q^2y}{qv^2}\right),$$

respectively.

W. A. MANNING,
Secretary of the Section.

PROJECTIVE DIFFERENTIAL GEOMETRY.

AN ABSTRACT OF FOUR LECTURES DELIVERED AT THE NEW HAVEN COLLOQUIUM, SEPTEMBER 5-8, 1906.

BY PROFESSOR E. J. WILCZYNSKI.

THESE four lectures were devoted to an exposition of the principal results belonging to the subject of projective differential geometry. The place of this subject in a systematic treatment of geometry is indicated by the following discussion.

A first important basis for the classification of the various geometries is furnished by the group concept. There is metric geometry, projective geometry, the geometry of the birational transformations, to mention only the most important. Together with this classification by means of the characteristic groups, there is the distinction between differential and integral geometry. The differential properties of a geometric configuration merely depend upon the fact that in a certain, perhaps very small, region, certain conditions of continuity are satisfied, that derivatives of a certain order exist, etc. These differential properties are studied by means of the differential calcu-

lus and do not depend upon the nature of the configuration (curve, surface, etc.) as a whole, but merely upon its character in a certain neighborhood. Opposed to the differential, we have the integral properties which depend upon the configuration as a whole. Thus within each geometry, characterized by its group we may still distinguish two divisions; we have differential and integral metric, differential and integral projective geometry; the geometry of birational transformations is likewise composed of two such divisions. Integral projective and differential metric geometry have been systematically developed. It is noticeable that the same is not true of differential projective or of integral metric geometry. In the nature of things differential geometry is more easily accessible than integral geometry. For the relation between the two is precisely that of the differential to the integral calculus. From the properties of differential geometry those of integral geometry may be obtained by integration, a process which requires the invention of special methods for every particular case. If, therefore, we can speak of an integral projective geometry as existing, it is nevertheless only in a very special sense. In fact, the configurations of projective geometry have, in almost all cases, been assumed to be algebraic, a restriction which is equivalent to an a priori integration. Leaving the algebraic cases aside, it is clear, therefore, that integral geometry, taken in the general sense, must be preceded by differential geometry. The metric half of this latter subject has been occupying the attention of mathematicians since the days of Monge and Gauss, and in the hands of their successors has reached a high degree of perfection. The same cannot be said of projective differential geometry. These lectures give an account of the first systematic treatment of a part of this subject, namely the projective differential geometry of curves and ruled surfaces.

The theory of curves in a space of $n-1$ dimensions is based upon the theory of the invariants and covariants of a linear homogeneous differential equation of the nth order, a point of view due to Halphen and developed by him in an admirable series of papers. The theory of ruled surfaces is based upon the consideration of a system of linear differential equations of the form

$$(A)\qquad \begin{aligned} y'' + p_{11}y' + p_{12}z' + q_{11}y + q_{12}z &= 0, \\ z'' + p_{21}y' + p_{22}z' + q_{21}y + q_{22}z &= 0. \end{aligned}$$

In connection with this system, consider the infinite group G of transformations

$$(G) \qquad \eta = a(x)y + \beta(x)z, \quad \zeta = \gamma(x)y + \delta(x)z, \quad \xi = f(x),$$

where a, β, γ, δ, f are arbitrary functions of x. The transformations of G convert (A) into another system of the same kind. The problem presents itself: to find the invariants and covariants of the system (A) under the transformations of the group G. This problem has been completely solved.

Let (y_i, z_i), $(i = 1, 2, 3, 4)$ be four pairs of simultaneous solutions of system (A). Then it may be shown that the most general pair of solutions is given by the equations

$$y = \sum_{i=1}^{4} c_i y_i, \quad z = \sum_{i=1}^{4} c_i z_i$$

involving four arbitrary constants $c_1, \cdots c_4$, provided that the determinant

$$D = \begin{vmatrix} y_1' & y_2' & y_3' & y_4' \\ z_1' & z_2' & z_3' & z_4' \\ y_1 & y_2 & y_3 & y_4 \\ z_1 & z_2 & z_3 & z_4 \end{vmatrix}$$

is not identically equal to zero. Four such pairs of functions, whose determinant does not vanish, and which satisfy system (A), constitute a fundamental system of simultaneous solutions. Conversely any four pairs of this kind determine a system of form (A) of which they constitute a fundamental system.

Interpret $(y_1, \cdots, y_4)$ and $(z_1, \cdots, z_4)$ as the homogeneous coordinates of two points p_y and p_z. As the independent variable x changes, p_y and p_z describe two curves C_y and C_z, the integral curves of system (A). There is a definite correspondence between the points of these curves, those being corresponding points which belong to the same value of x. Join corresponding points p_y and p_z by a straight line h_{yz}. The ruled surface S which is the locus of these lines is called the integrating ruled surface of system (A). It is easy to show that any non-developable ruled surface may be defined by a system of form (A); the developables are excluded by the condition that the determinant D shall not be equal to zero.

The transformations of the group G leave the ruled surface S invariant. In fact, their geometric significance is merely to replace the two curves C_η and C_ζ upon S by two other curves C_η and C_ζ upon S, the point-to-point correspondence being again determined by the generators of S. The transformations of G further involve the most general change of the independent variable. An invariant of system (A) under the transformations of the group G therefore has a significance for the ruled surface S which is independent of the particular curves upon S chosen as fundamental curves, and of the choice of the independent variable. Moreover this significance is of a projective nature, since any projective transformation of the surface S gives rise to the same system of form (A), and since the most general integrating ruled surface of (A) is a projective transformation of any particular one.

Upon these considerations the lecturer based his theory of ruled surfaces which, together with some known results, contains a great variety of new ones. It is impossible to give an intelligible account of the details of this theory without greatly exceeding the space at our disposal. It is expounded in detail in the lecturer's recent treatise.* This brief abstract will suffice to indicate the character of the subject under investigation and the nature of the methods employed.

ON LOCI THE COORDINATES OF WHOSE POINTS ARE ABELIAN FUNCTIONS OF THREE PARAMETERS.

BY PROFESSOR J. I. HUTCHINSON.

(Read before the American Mathematical Society, September 3, 1906.)

A particular case of a surface the homogeneous coordinates of whose points are theta functions of three variables u_1, u_2, u_3 (connected by a relation $\vartheta = 0$) is given by Humbert† and studied in considerable detail by means of this parametric rep-

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* E. J. Wilczynski. Projective differential geometry of curves and ruled surfaces. Leipzig, B. G. Teubner, 1906.

† "Sur une surface du sixième ordre liée aux fonctions abéliennes de genre trois," *Liouville*, 1896, pp. 263–293.

resentation. Such a surface may be regarded as a section of a spread S (or three-dimensional locus) in space of four dimensions the coordinates $x_0, \cdots, x_4$ of whose points are five theta functions $\Theta_i(u_1, u_2, u_3)$ of the same order m and characteristic (c). It is the partial, or complete intersection of S by a flat space of three dimensions, a flat.

For brevity let the symbol $\omega_k (k = 1, \cdots, 64)$ be used to refer to any one of the 64 systems of half-periods. The number of conditions that Θ_i vanish of order r_k for the half-period ω_k is in general $\frac{1}{6} r_k(r_k + 1)(r_k + 2)$. The function then contains

$$m^3 - \tfrac{1}{6} \sum_{k=1}^{64} r_k(r_k + 1)(r_k + 2)$$

linear constants. The cases in which this number reduces to 5 are of particular interest since all such functions are linearly expressible in terms of the x_i.

The degree n of S is the number of non-congruent solutions of three simultaneous equations of the form $\sum a_i x_i = 0$, which is

$$n = 6m^3 - \sum r_k^3.$$

Combining this with the relation

$$m^3 - \tfrac{1}{6} \sum r_k(r_k + 1)(r_k + 2) \geqq 5$$

we have

$$n - 30 \geqq \sum (3r_k^2 + 2r_k).$$

Since the right member is positive, it follows that n is not in general less than 30, and for a given n there are only a finite number of cases to be considered.

If, for example, we take the functions $x_1, \cdots, x_4$ employed by Humbert (l. c., page 270) and assume $x_0 = \vartheta_1 \vartheta_2$, we find S to be of degree 36, since the coordinate functions vanish to the first order for 12 half-periods. Corresponding to these half-periods are 12 planes lying completely in S. The section of S by the flat $x_0 = 0$ consists of these 12 planes, each counted twice, and two sextics $\vartheta_1 = 0$, $\vartheta_2 = 0$, the first of which is the one discussed by Humbert. The second can be transformed linearly into the first, since ϑ_1 and ϑ_2 permute by the addition of a suitable half-period.

If the x_i are all even (or odd), the above relations are to be modified, since to the two different sets of values u_h, $-u_h$ of the

parameters corresponds only one point x_i. Denote by $2s_k$, $2t_l + 1$ the order of vanishing at a half-period when that order is even or odd respectively. Then

$$6m^3 - \sum_{k=1}^{\kappa} 8s_\kappa^3 - \sum_{l=\kappa+1}^{64} (2t_l + 1)^3 = 2n$$

and the number of linear constants in the x_i is

$$\tfrac{1}{2}(m^3+\mu)-\tfrac{1}{6}\sum_{k=1}^{\kappa} s_k(s_k+1)(4s_k-1)-\tfrac{1}{6}\sum_{l=\kappa+1}^{64} t_l(t_l+1)(4t_l+5) \geqq 5$$

in which κ and μ have the following values:

1. $\kappa = 64$, $\mu = 8$; m even, Θ even, characteristic zero;
2. $\kappa = 0$, $\mu = -8$; m even, Θ odd, characteristic zero;
3. $\kappa = 32$, $\mu = 0$; m even, Θ odd, characteristic zero;
4. $\kappa = 36$, $\mu = -1$; m odd, Θ $\left\{\begin{smallmatrix}\text{even}\\ \text{odd}\end{smallmatrix}\right\}$, characteristic $\left\{\begin{smallmatrix}\text{odd}\\ \text{even}\end{smallmatrix}\right\}$;
5. $\kappa = 28$, $\mu = 1$; m odd, Θ $\left\{\begin{smallmatrix}\text{even}\\ \text{odd}\end{smallmatrix}\right\}$, characteristic $\left\{\begin{smallmatrix}\text{even}\\ \text{odd}\end{smallmatrix}\right\}$.

As in the general case, we deduce the condition

$$n + 3\mu + 2 - \tfrac{1}{2}\kappa \leqq \sum_{k=1}^{\kappa} (3s_k^2 - s_k) + \sum_{l=\kappa+1}^{64} (3t_l^2 + 2t_l),$$

which gives a restricted number of choices for m, s_k, t_l when n is given. The least value that n can have appears to be 12. This occurs in the two following cases:

I. Let α, β, γ denote three different systems of half-periods, and (α), (β), (γ) represent their characteristics. These can be taken odd without loss of generality. For, if (c_1), (c_2), (c_3) denote any three characteristics, assume (g) so that $(c_1 + g) = (\alpha)$, $(c_2 + g) = (\beta)$, $(c_3 + g) = (\gamma)$ in which the right members are odd. Then we have the three groups *

$$(g_1) = (c_2 + c_3) = (\beta) + (\gamma),$$
$$(g_2) = (c_3 + c_1) = (\gamma) + (\alpha),$$
$$(g_3) = (c_1 + c_2) = (\alpha) + (\beta).$$

Since any three groups either contain four characteristics (α),

* In the sense of Weber, Abel'sche Functionen, p. 19.

(β), (γ), (δ) in common, or any two have six in common (none of them paired), it follows that (α), (β), (γ) can be chosen in a variety of ways. The sum $(\alpha + \beta + \gamma)$ is either odd, (δ), or even. In the first case the theta functions ϑ_0, ϑ_1, ϑ_2, ϑ_3 of the first order and characteristics (0), (g_1), (g_2), (g_3) respectively vanish simultaneously for the half-periods α, β, γ. There are five linearly independent theta functions with $m = 2$ and characteristic (0) which vanish of order 2 for α, β, γ. We may take for these $x_i = \vartheta_i^2 (i = 0, \cdots, 3)$, $x_4 = \Theta_4$, the last being any one linearly independent of the other four. The points x_i lie on a spread S of degree 12, which has 61 quadruple points corresponding to the half-periods other than α, β, γ.

By equating to zero a theta function of the form $\Theta = \sum a_i x_i$ we obtain a surface of degree 12 in ordinary space. There are various types of such, according to the number and arrangement of the nodes of S which lie in the given flat. There are only four cases in which such a surface degenerates, namely, when $\Theta = x_i (i = 0, \cdots, 3)$. Then it becomes a sextic T counted double. The sextic, counted singly, may be represented by $\vartheta_i = 0$. There are 3 singular tangent planes and 25 double points, 8 in each of the singular planes, and one at their common point. By equating to zero other theta functions of the first order we obtain 24 twisted quartics of genus zero, 24 quintics, and 12 sextics lying on T.

II. If $(\alpha + \beta + \gamma)$ is even the groups

$$(g_1) = (\beta) + (\gamma) = (\sigma_1') + (\sigma_1'') = \cdots = (\sigma_5') + (\sigma_5''),$$
$$(g_2) = (\gamma) + (\alpha) = (\sigma_1'') + (\sigma_1) = \cdots = (\sigma_5'') + (\sigma_5),$$
$$(g_3) = (\alpha) + (\beta) = (\sigma_1) + (\sigma_1') = \cdots = (\sigma_5) + (\sigma_5'),$$

in which (σ_i), (σ_i'), (σ_i'') are odd, are necessarily of the form indicated, any two having six characteristics common. Otherwise they would all have four in common, whence it would follow $(\alpha) + (\beta) = (\gamma) + (\delta)$, (δ) being odd, which is contrary to the hypothesis.

The six theta functions with characteristics (0), $(\alpha + \sigma_i) = (\beta + \sigma_i') = (\gamma + \sigma_i'') (i = 1, \cdots, 5)$ vanish simultaneously for the half-periods α, β, γ.* These will be denoted by ϑ_0, ϑ_1,

* It is evident that the three characteristics (α), (β), (γ) occur simultaneously in the five groups $(g_i) = (\alpha + \sigma_i)$ and only in these. This is a more precise form of Weber's theorem — "Irgend drei ungerade Charakteristiken, deren Summe gerade ist, kommen immer in mehreren Gruppen vor," *loc. cit.*, p. 22.

$\cdots$, ϑ_3 respectively. Then $x_i = \vartheta_i^2 (i = 0, 1, \cdots, 4)$ determine a spread S' of degree 12 having 61 quadruple points. As in the preceding example, to each of the half-periods α, β, γ corresponds a two-spread (a two-dimensional locus) of degree 4.

Since two functions ϑ_i vanish simultaneously for 12 half-periods, three of which are α, β, γ, it follows that the flats $ax_i + bx_j = 0$ intersect S' in 9 of its nodes. If we require it to pass through a tenth not lying in either x_i or x_j, there are 29 points to choose from, and hence we determine $29.15 = 435$ surfaces of degree 12, each having 10 quadruple points.

Every flat of the form $ax_i + bx_j + cx_k = 0$ contains 3 nodes of S' and can be made to pass through two other nodes by properly choosing the constants. There are thus determined $20.45 = 900$ surfaces of degree 12 which have 5 quadruple points.

The coördinate flats $x_i = 0$ contain doubly covered sextics. Consider, for example, the sextic $\vartheta_0 = 0$, which we denote by T'. It has 25 nodes, 5 cubic curves $\vartheta_i = 0 (i = 1, \cdots, 5)$ lying in singular tangent planes, 18 quartic (genus zero), 30 quintic, and 10 sextic twisted curves.

CORNELL UNIVERSITY,
August, 1906.

ASSOCIATED CONFIGURATIONS OF THE CAYLEY-VERONESE CLASS.

BY DR W. B. CARVER.

(Read before the American Mathematical Society, September 3, 1906.)

In this paper, S_n will be used to denote a flat space of n dimensions; and the notation $C^r_{n,r}$ and the words "chiastic," "copoint," etc., will be used in the sense defined in the author's earlier paper * on these configurations.

Given five planes in S_3, we can, in general, construct the polar point P of any one of the five with respect to the 4-point determined by the other four planes. The five points so constructed, when joined by lines and planes, give a complete 5-point chi-

* "On the Cayley-Veronese class of configurations." *Transactions Amer. Math. Society*, vol. 6, pp. 534–545 (October, 1905).

astic with the complete 5-plane; and the 5-point and 5-plane are said to be *associated.**

Projecting this entire figure upon a plane (by what Veronese † calls an *univoca*, and his German translators an *eindeutige* projection) we have the following:

A $C^3_{5,2}$ (the Desargues configuration of two perspective triangles with their center and axis of perspective) breaks up in five ways into a $C^3_{4,2}$ (a complete 4-point) and a $C^2_{4,2}$ (a complete 4-line). Each $C^2_{4,2}$ and the corresponding $C^3_{4,2}$ determine a point P, which may be called the polar point of the $C^2_{4,2}$ with respect to the $C^3_{4,2}$; and the five points so determined, when joined by lines, form a $C^4_{5,2}$ (a complete 5-point) chiastic with the $C^3_{5,2}$. This $C^2_{5,2}$ may be said to be associated with the $C^4_{5,2}$.

This simple conception may be doubly extended, and we obtain the following general theorem:

Let there be given, in S_r, a $C^n_{n+2,r}$. This $C^n_{n+2,r}$ breaks up in $n+2$ ways into a $C^n_{n+1,r}$ and a $C^{n-1}_{n+1,r}$, and we can construct the polar point of each $C^{n-1}_{n+1,r}$ with respect to the corresponding $C^n_{n+1,r}$. The $n+2$ points so constructed give a $C^{n+1}_{n+2,r}$ chiastic with the $C^n_{n+2,r}$, and we may speak of the two as being associated.

It remains now to give a method for constructing the polar point of the $C^{n-1}_{n+1,r}$ with respect to the $C^n_{n+1,r}$. This can be done, without going outside of the space S_r in which the configurations lie, in the following manner:

Let the $C^n_{n+1,r}$ be lettered with $n+1$ letters, as described in the author's earlier paper. Any $n-1$ letters denote a line of the $C^n_{n+1,r}$ and on this line lies a point of the $C^{n-1}_{n+1,r}$. The point of the $C^{n-1}_{n+1,r}$ has a polar point P_{n-1} with respect to the two points of the $C^n_{n+1,r}$ lying on that line. (Let this P_{n-1} have a subscript made of the $n-1$ letters belonging to the line.) Any $n-2$ letters denote a triangle, or $C^2_{3,r}$, of the $C^n_{n+1,r}$. If we join the P_{n-1} on each line of such a $C^2_{3,r}$ to that point whose symbol contains the $n-2$ letters of the $C^2_{3,r}$ but not the remaining letter of the P_{n-1}, the three such joins meet in a point P_{n-2}. In the $C^n_{n+1,r}$ $n-3$ letters denote a $C^3_{4,r}$. If we join the P_{n-2} of each $C^2_{3,r}$ in such a $C^3_{4,r}$ to the proper point (de-

* Cf. Morley, "Projective coordinates," *Transactions Amer. Math. Society*, vol. 4. See p. 292.

† Cf. Grundzüge der Geometrie, p. 614.

termined as before), these four joins meet in a point P_{n-3}. * * * Finally, if we join each P_1 to that point of the $C^n_{n+1,r}$ whose symbol does not contain the letter of the P_1, these $n+1$ joins meet in a point P which is the polar point of the $C^{n-1}_{n-1,r}$ with respect to the $C^n_{n+1,r}$.

The dual theorem for S_r gives a polar copoint, or S_{r-1}, of a $C^{r+1}_{n+1,r}$ with respect to a $C^r_{n+1,r}$; and, when $r=2$, we have, in the plane, a polar line of a $C^3_{n+1,2}$ with respect to a $C^2_{n+1,2}$. This special case in the plane is treated by Caporali, but his entire method is different from the method used in this paper.

If, having constructed in S_3 the polar point P of a plane with respect to the 4-point determined by four other planes, we project the whole figure upon a plane, *using P as the center of projection*, we obtain a $C^3_{4,2}$ and a $C^2_{4,2}$ which are associated, and the polar point of the former with respect to the latter is evidently indeterminate. In general, suppose we take, in S_n, the polar point P of a copoint, or S_{n-1}, with respect to the $(n+1)$-point determined by $n+1$ other copoints, and then project this figure upon an $S_r (r<n)$ from a co-S_r, or S_{n-r+1}, taking an S_{n-r+1} incident with the point P. We will thus obtain in S_r a $C^n_{n+1,r}$ and a $C^{n-1}_{n+1,r}$ which are associated; and the polar point of the $C^{n-1}_{n+1,r}$ with respect to the $C^n_{n+1,r}$ is indeterminate. *Whenever* a $C^n_{n+1,r}$ and a $C^{n-1}_{n+1,r}$ are associated, the polar point of the latter with respect to the former is indeterminate. This fact also Caporali notes for the plane case.

This polar idea for configurations admits of one more extension. If we have, in S_n, a $C^{n-k}_{n+1,r}$ chiastic with a $C^n_{n+1,r}$ $(k \leqq r-1)$,* we may define the polar S_{k-1} of the $C^{n-k}_{n+1,r}$ with respect to the $C^n_{n+1,r}$ thus:

There are ∞^{k-1} $C^{n-1}_{n+1,r}$'s chiastic † with the $C^n_{n+1,r}$ and also with the $C^{n-k}_{n+1,r}$, and any k of these $C^{n-1}_{n+1,r}$'s are sufficient ‡ to determine the $C^{n-k}_{n+1,r}$. Construct the polar point of each of k of these $C^{n-1}_{n+1,r}$'s with respect to the $C^n_{n+1,r}$ and these k polar points will lie upon an S_{k-1} which is the polar S_{k-1} of the $C^{n-k}_{n+1,r}$ with respect to the $C^n_{n+1,r}$.

There is of course a dual theorem which defines a polar S_{r-k} of a $C^{r+k}_{n+1,r}$ with respect to a $C^r_{n+1,r}$.

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* Author's paper, *loc. cit.*, paragraph under theorem II.

† *Loc. cit.*, theorem VI.

‡ *Loc. cit.*, theorem V.

VON HELMHOLTZ.

Hermann von Helmholtz. Von L. KOENIGSBERGER. 3 Bde., 8vo. Braunschweig, Vieweg und Sohn. Bd. 1: 1902, xi + 375 pp., 3 portraits; Bd. 2: 1903, xiv + 383 pp., 2 portraits; Bd. 3: 1903, ix + 142 pp., 4 portraits.

Vorlesungen über theoretische Physik. Von H. VON HELMHOLTZ. 6 Bde., large 8vo. Leipzig, J. A. Barth. Bd. 1_1: *Einleitung zu den Vorlesungen über theoretische Physik*, hrg. von KÖNIG und RUNGE, 1903, 50 pp., 1 portrait; Bd. 1_2: *Vorlesungen über die Dynamik discreter Massenpunkte*, hrg. von KRIGAR-MENZEL, 1898, x + 380 pp.; Bd. 2: *Dynamik continuirlich verbreiteter Massen*, hrg. von. KRIGAR-MENZEL, 1902, viii + 247 pp.; Bd. 3: *Vorlesungen über mathematische Principien der Akustik*, hrg. von KÖNIG und RUNGE, 1898, x + 256 pp.; Bd. 4: *Elektrodynamik*, "soll bald erscheinen"; Bd. 5: *Vorlesungen über die elektromagnetische Theorie des Lichtes*, hrg. von KÖNIG und RUNGE, 1897, xii + 370 pp.; Bd. 6: *Vorlesungen über Theorie der Wärme*, hrg. von F. RICHARZ, 1903, xii + 419 pp.

HELMHOLTZ died on September 8, 1894. Although he was at that time considerably past seventy years of age, he was still in the midst of his researches, still as enthusiastic as ever, still as piercing of intellect. On July 9, just before his last illness, he had been busy correcting the proof of his pupil Hertz's Mechanics, but nevertheless had stated that it had been a lucky day for him because he had discovered what he and many others before him had long been looking for. What this was we shall never know; for before he had had the necessary time to write the matter out and present it to the Academy he was mortally ill. His own researches and those of his pupil Hertz, who had died so shortly before, were by no means the only things that occupied his mind. He was director of the Reichsanstalt and had come to the decision two years previous to publish his lectures on mathematical physics. Indeed the fifth volume, on the electromagnetic theory of light, was already in the press with the proofs two-thirds read. This was a fitting end to the activity of one who for fifty years has been astonishing the world with the fecundity of his investigations

in widely different fields of science. As Lord Kelvin said in an address before the Royal society in November 1904: "Of the whole of Helmholtz's great and splendid work in physiology, physics, and mathematics, I doubt whether any one man may be qualified to speak with the power which knowledge and understanding can give."

Upon some of Helmholtz's pupils devolved the task of seeing his lectures through the press; and at the deathbed of the great physicist's youngest son, when the name of von Helmholtz was slipping away from the world forever, his old friend and long time colleague at Heidelberg, Koenigsberger, determined to write his biography. The biography and the lectures (with the exception of the fourth volume) are now complete; and with the books which Helmholtz himself published they form a readily accessible monument which some of us may believe a more impressive memorial to his genius than that unveiled amid such royal pomp in the garden of the University of Berlin.

Koenigsberger's biography is not merely an account of Helmholtz's private life, it is not an analysis of his scientific achievements, it is not the memoirs of an intimate acquaintance nor the critical history of an outsider; it is all of these together. On one page the reader will find some family correspondence, perhaps concerning the health of Robert or Fritz; on the next, will be a critique of a series of articles on the foundations of geometry or sometimes an unprinted public address. As one reads, he follows the whole life of the master.

The work is divided into sections corresponding to the different positions which Helmholtz occupied — a student, a naval surgeon, teacher in an academy of fine arts, professor of physiology at Königsberg, professor of anatomy at Bonn, of physiology at Heidelberg, of physics at Berlin, and president of the Reichsanstalt. The titles alone should seem to justify Kelvin's remark. It should be remembered that Helmholtz did not have a free choice as to his profession. He always said that he was a born physicist. But his father was poor; being a physicist was too dear; the boy must choose medicine as his work even though both he and his father regretted it. The result was one of the world's greatest physiologists, with the Lehre von den Tonempfindungen and the Physiologische Optik as chief witnesses. He could not have been a greater physicist even if he had not been a physiologist, perhaps not so great.

It would be ungracious to draw any moral concerning the elective system of studies and the necessity of doing what one pleases too early in life.

Helmholtz had a very marked ability for keeping out of ruts. A predisposition for getting snugly into some rut is perhaps one reason for the opinion current in some quarters to the effect that one's work is done at the advanced age of forty. If Helmholtz had lived no longer, we should have had the Erhaltung der Kraft, the Integrale der hydrodynamischen Gleichungen welche den Wirbelbewegungen entsprechen, and the Handbuch der physiologischen Optik; but what of the Lehre von den Tonempfindungen and indeed the last third of the Physiologische Optik, what of the work on the foundations of geometry which led to Lie's researches on that subject, and the investigations on electromagnetism which culminated in Hertz's discoveries, or the Thermodynamik chemischer Vorgänge which, written at the age of sixty, would have been the great landmark in physical chemistry if Gibbs's then unknown work had not appeared half a dozen years before? And this is not all. It is necessary to read the whole biography to appreciate how Helmholtz retained his spontaneity and power into the last days of his life. A rare parallel to this is seen in the life of his close friend, Kelvin.

In its general make-up as a book the biography is almost de luxe. The paper is heavy and the type, which fortunately is Roman, is clear. The reproductions of portraits, for the most part by Lenbach, add to the artistic appearance of the work, and to its personal interest. The scientific world has reason to thank the publisher for the form in which the volumes have appeared just as it should be very grateful to the author who, though by no means a young man, has devoted so much time to accomplish religiously his self-appointed task.

Helmholtz's Lectures on theoretical physics are particularly interesting and instructive because he himself was personally responsible through his own researches for so large a part of their contents. It is this which gives them their life and individuality. Moreover the acquaintance of the author with so great a realm of physics, mathematics, and biology gives a virility to his lectures and a powerful variety to his illustrations which it would be difficult to find elsewhere.

The introduction of fifty pages which forms the content of the first part of the first volume is a sort of epistemology or

methodology of the physical sciences. This subject was of great interest to Helmholtz and upon it he had printed a large number of papers which may be found admirably analyzed in the biography. The author was too much of a physiologist to be won over without restriction to that abstract formalism which was represented by Thomson and Tait and by Kirchhoff. He doubtless could understand their point of view when they undertook to demonstrate that force did not exist; but he preferred to retain force as a fundamental idea. In the treatment of the number system the same practical point of view appears. Note the statement: "That diagonals of squares exist in nature there is no question; and so, too, definite physical investigation may thrust other irrationalities upon us. The difficulty or the impossibility arises here, not in regard to the reality of such quantities or ratios, but in the incompleteness of our methods. . . . Similarly with the much discussed question of the existence of continuous quantities." This and other statements which the author makes may furnish a sounder basis for metaphysical speculation than many a more complete theory developed by those less in touch than he was with the real living world.

Such careful attention to the underlying conceptions is everywhere typical of the lectures. The discussion of the dynamics of discrete masses begins with a detailed account of such things as the material point, coördinates, continuity and differentiability, velocity, acceleration, Newton's laws, and so forth. It would be difficult to find a clearer and more careful statement of these fundamental matters. As might be expected from the director of the Reichsanstalt, the theory of pendular vibrations is treated in great detail, sixty-five pages being given to it. To illustrate the way in which the author gives the student the necessary mathematical methods as he goes along we may mention that the questions of least squares and of linear non-homogeneous differential equations of the second order find their treatment at the points where each is needed. Stress is laid on the importance of the general integrals of motion, the integrals of momentum, of moment of momentum, and of energy. Work and energy receive the attention which one might expect to be given them by the author of the Erhaltung der Kraft. The various fundamental principles such as that of virtual velocities, of d'Alembert, and of Hamilton, and the lagrangian equations of motion are all taken up. It may

be noted that Appell in his Mécanique rationnelle follows this same plan of introducing into dynamics at a somewhat early stage these higher principles. It would be difficult to find an elementary treatment of mechanics which could better serve as an introduction to mathematical physics than this of Helmholtz.

Before taking up the theory of continua, the author gives a short discussion of the relation of the theory to the molecular hypothesis, and of the difference between ordered and unordered motion in the interior of the continuum. He then establishes a connection with the homogeneous strain by developing the displacement into a Taylor's series and examining the effect of the terms of higher order. After a preliminary treatment of strain, he makes the statement that the internal forces in a continuum follow the law of action and reaction and hence are conservative. This leads the way to Green's method of developing the theory of elastic bodies from the existence of a potential for the forces of deformation. The analytic theory of elasticity is beset with massive formulas. These the author largely avoids by turning his attention to the more physical side of the question. He sets about determining the function whose space integral gives the potential energy in a variety of different types of elastic bodies. The following section of the book treats of the equilibrium of elastic bodies. Those cases are solved which are of use in actually determining the elastic constants of the bodies. Finally the question of the propagation of plane or spherical waves, whether longitudinal or transversal, is taken up and solved. Helmholtz was about seventy-three years old when he prepared and delivered these lectures and must have been, if the lectures may be taken in evidence, an admirable teacher; for he shows a genius in avoiding analytic difficulties wherever he can, and in resolving them clearly for the student when to avoid them is impossible or undesirable.

With the third volume, which is on sound, there begins to appear a considerable amount of repetition. The amount of attention paid to the motion of a pendulum was noted in speaking of the first volume; here the matter must needs come up again. So different, however, is the method of procedure and so much is the original problem generalized that the repetition seems advantageous rather than otherwise. It is a serious mistake to avoid altogether the repetition of matters and methods which are so fundamental that they should become second nature

to the student. It is not possible to be old friends at first sight. Of course Fourier's series are the foundation of a large part of the mathematical work. Instead of assuming that they are known or may be learned by reference to some standard text, the author patiently and in all detail develops the method of Fourier's series, enters upon the necessary proofs of convergence and even goes so far as to show that the convergence is toward the right value. This development of the mathematical algorithms as they are needed goes a long way toward giving the reader that confidence which is so important. In treating waves in space, Green's theorem and Huyghens's principle are taken up in the same manner. In discussing vibrations, each different type, whether a plucked string, a stretched string, a membrane, free air, or air confined to a tube, is developed separately and in natural order.

The fourth volume, on electrodynamics, has been announced as "soll bald erscheinen" for almost a record breaking number of years. It is even rumored that the volume may never appear. This is probably due to the great changes which have taken place in electromagnetic theory since the lectures were delivered. The editors may feel that there would be little use in printing a book which would be out of date before it was written. It seems to us that this reason should not prevail here. Helmholtz's researches on electricity and magnetism were by no means insignificant. A large part of the time of his later years was spent in this field of physics. It was he who developed the theory of an ether which included Maxwell's as a special case. His lectures on electrodynamics could not fail to be of interest as a matter of history if in no other way. Furthermore it is inconceivable that what Helmholtz might have to say on the subject, his methods of attack, and his points of view should not be highly important and instructive even after the world has moved somewhat away from his position. We hope the volume may still appear.

The fifth volume, that on the electromagnetic theory of light, was the first to be published. Fortunately it is independent of the preceding one. With characteristic straightforwardness an introduction on some points in electrical theory and on waves in a continuous medium comes before the introduction of the Maxwell equations. In their proper places the elements of the theory of the potential function, Green's theorem, plane and spherical waves, and Huyghens's principle reappear, although

each has been taken up at least once before. After a long presentation of diffraction and the establishment of the laws of geometric optics from the fundamental dynamical equations, there follows a detailed treatment of the customary questions of geometric optics. Polarization and dispersion are not neglected; but the fact that the work is one on general physical optics rather than on special mathematical problems in optics is clearly indicated by the meagre attention paid to crystalline media — only eight pages. This is but another instance of the author's avoidance of complicated mathematics and insistence upon that which will be of most value to the practical physicist.

The lectures on the theory of heat are a charming close to the whole series. The introduction on "warmness" and temperature reminds one irresistibly of the author's philosophical speculations with which the series began. The treatment of problems in conduction with the repetition of the necessary mathematical tools, Green's theorem and Fourier's series, not omitting even the proofs of convergence, recalls the general style of the other volumes. The applications of these results are, however, new and lead up to the introduction of Fourier's integral with a full explanation of what this new tool does for one. Throughout the volume, the illustrative examples have even more than their accustomed variety — cooling of the earth, diurnal and annual variations in subterranean temperature, the vital theory of the black body, and Kirchhoff's law. The next large division of the volume concerns thermodynamics. As might be expected, the underlying concepts, the first law or conservation of energy, the second law or dissipation of energy, the entropy, the "free" energy, and so on, are set forth with especial pains. It would be difficult for the reader to fail to get the right idea. It is here and in the following section on cyclic systems that we find some account of the researches which took the major part of the author's free time after passing the age of sixty. The work appears to be still new to him. There is a freshness about the style which is not so much in evidence in the volumes which deal with his earlier work.

It has not been our intention here to give an account of Helmholtz's life and works on mathematical physics — this were too great a task — but to indicate as briefly as might be what may be found in the eight or nine volumes which treat of these subjects and which appeared during the decade after his

death. Of the lectures Koenigsberger says: "(Sie) bilden jetzt die ausgezeichnetsten Lehrbücher einzelner Theile der mathematischen Physik." Nothing could be truer. And may we not in all seriousness ask the question whether these live works of a master genius may not be more inspiring and really better text-books for the student than the work of some lesser person, no matter how careful and industrious he be? Of the biography let us add that it ends with consummate inspiration in the words which Helmholtz himself applied to Goethe and Beethoven: "Wir verehren in ihnen einen Genius, einen Funken göttlicher Schöpferkraft, welcher über die Grenzen unseres verständig und selbstbewusst rechnenden Denkens hinausgeht. Und doch ist der Künstler wieder ein Mensch, wie wir, in welchem dieselben Geisteskräfte wirken wie in uns selbst, nur in ihrer eigenthümlichen Richtung reiner, geklärter, in ungestörterem Gleichgewichte, und indem wir selbst mehr oder weniger schnell und vollkommen die Sprache des Künstlers verstehen, fühlen wir, dass wir selbst Teil haben an diesen Kräften, die so Wunderbares hervorbrachten."

EDWIN BIDWELL WILSON.

YALE UNIVERSITY, NEW HAVEN, CONN.,
October 8, 1906.

PIERPONT'S THEORY OF FUNCTIONS.

The Theory of Functions of Real Variables, Volume I. By JAMES PIERPONT, Professor of Mathematics in Yale University. Boston, Ginn and Company, 1905. xii + 560 pp.

SINCE the time of Weierstrass the so-called rigorous style in mathematical writing has increased constantly in favor until in recent years it has become a commonplace instead of a rarity. Such myriads of microscopic ϵ's and δ's have penetrated our mathematical thinking that it would be impossible to rid the system of them entirely even if it were desirable to do so. A return to the externals of the intuitive style of writing would, however, enable any one individual to obtain a conception of a much wider range of investigations, and it seems possible that such a return may be made after mathematicians have come to an understanding acknowledged by all of the precise meaning

of the various notions of continuity and limits used in analysis. Many of the details now necessary in a rigorous exposition could then be omitted. Before this can happen, however, books of the general character of the one with which this review is concerned must become familiar reading.

Professor Pierpont's book is the first of two volumes in which he will present the subject matter of his lectures at Yale University on advanced calculus and the theory of functions of real variables. The contents may be roughly divided into three parts, the first on the real number system and the real exponential and logarithmic functions, the second on differential calculus with chapters on point aggregates and implicit functions, and the third on integration, including a more than usually complete discussion of improper integrals and integrals involving a parameter. The purpose which the author had in mind is made clear in one of the paragraphs of the preface. After a statement that the American student about to enter into graduate work often understands much of the formal processes of the calculus without having a clear knowledge of the conditions under which their applications are justifiable, he writes:

"The problem therefore arises to examine carefully the conditions under which the theorems and processes of the calculus are correct and to extend as far as possible or useful the limits of their applicability."

Some of the chapters deserve especial mention on account of their clearness and finish. For example, it would be hard to find more attractive accounts of the theory of implicit functions, indeterminate forms, and the definition and properties of a definite integral than are given in Chapters IX, X, and XII, respectively. On the other hand other parts of the book impress one as being relatively unfinished and somewhat overcomplicated. Among these are the sections devoted to integrals involving a parameter, which will be mentioned later. The pedagogical principle that clearness is attained by proving special cases of theorems first, instead of treating them as corollaries to the general case, has caused the book to be larger than it would otherwise have been, and it is doubtful if after all the advantage gained is very great.

Among the most interesting features of the book are the sections entitled "Criticism." As the author states in the preface, nothing stimulates the student's critical sense more powerfully than asking him to detect the flaws in a piece of reasoning

which at an earlier stage of his training he considered correct. It is especially interesting, therefore, to find after each important discussion, sections in which the inaccuracies of the more elementary treatises are exposed. Mention may also be made here of the illustrative examples and the special theorems on limits. The former are numerous and well chosen, often being constructed to show how a theorem may fail to be true when its initial hypotheses are not satisfied. The latter occur especially in Chapters II, III, and VI, where many useful inequalities and limits are calculated.

In Chapters I and II Professor Pierpont has developed the theory of the real number system in a way which can not fail to be of interest. Supposing the theory of the system I of positive integers to be known, he introduces in succession three new systems, F, R, $\mathfrak{R}$, the first consisting of elements (a, a') with the properties of positive rational fractions, the second of elements $\{a, a'\}$ with the properties of the differences $a - a'$ of positive rational numbers, the third being the system of all real numbers. It is seen that the system I is closed only under addition and multiplication, while F is closed under addition, multiplication and division. R and $\mathfrak{R}$ are closed under all the rational operations and $\mathfrak{R}$ has the additional property of admitting limiting processes. Each new system consists of entirely new elements but in each there is a subsystem with elements and properties in correspondence with those of the one just preceding.

In Chapter II, the irrational number is defined after Cantor to be * a sequence of rational numbers

$$(a_1, a_2, a_3, \cdots, a_n, \cdots)$$

having the property that for any $\epsilon > 0$ there exists a positive integer m such that $|a_n - a_\nu| < \epsilon$ whenever n and ν are greater than m, or in Professor Pierpont's concise and satisfactory notation, a sequence for which

$$\epsilon > 0, m, \quad |a_n - a_\nu| < \epsilon, \quad n, \nu > m.$$

The totality of such sequences is the real number system $\mathfrak{R}$.

* Professor Pierpont writes on p. 36, "every regular sequence *defines a number.*" The difference between saying that the sequence *is* a number and saying that it *defines* a number, is slight. The former is perhaps less mysterious.

The sections (§§ 112–123) in which it is pointed out that the numbers of $\mathfrak{N}$ and the points of a line are in one-to-one correspondence are not very convincing because to make a logical demonstration of this fact it would be necessary to state explicitly the postulates which define a geometric continuum. As far as the applications in other parts of the book are concerned it would have sufficed to show that the points of a line can be used as a kind of notation for the real number system. Indeed much of graphic mathematics which seems intuitive would be precise if the graphs were regarded merely as notations for ideas elsewhere sharply defined. In the latter part of the chapter the definition of a Dedekind partition of the rational numbers is defined and the correspondence of these to the numbers of $\mathfrak{N}$ is shown.

Taken altogether, Professor Pierpont's treatment of the number system is a very suggestive one, especially valuable from the pedagogic standpoint. There are a few places where improvement seems possible. On page 24 one might infer that the properties 1°, 2° characterize a sequence. That this is not true is shown by the example *

$$a_1, a_2, a_3, \cdots, a_n, \cdots; b_1, b_2, \cdots, b_n, \cdots,$$

which has the properties referred to. These properties, however, have but slight importance in the following pages, as the notion of sequence actually used has always been a set of elements in one-to-one correspondence with the positive integers. In connection with the definitions of order and the algebraic operations on irrational numbers some essential points have been omitted. It should be shown (page 39) that any two numbers α, β satisfy one and but one of the relations $\alpha \lessgtr \beta$. Inasmuch as the same number has an infinity of different representations, the discussion is hardly complete without a proof that the order relations and the result of any algebraic operation are independent of the particular representations chosen for the numbers involved. Lack of space would render it inadvisable to give a complete treatment of the number system from the standpoint of the theory of assemblages, but in the sections in which the properties of R and $\mathfrak{N}$ are discussed it would have been easy to point out especially those which as postulates characterize these systems categorically.

* Huntington, "Types of serial order," *Annals of Math.*, 2d series, vol. 6, p. 169.

In Chapter III the theory of the exponential a^z is based upon the equation $x^n = a$, n a positive integer. This equation determines uniquely a Dedekind partition which defines a unique position root ρ, designated $a^{1/n}$. The definition of a^z when z is any rational number follows immediately, and when z is an irrational number $(r_1, r_2, \cdots, r_\nu, \cdots)$, a^z is the number $(a^{r_1}, a^{r_2}, \cdots, a^{r_\nu}, \cdots)$. Similarly the theory of the logarithm is based upon the equation $b^x = a$.

With Chapter IV begins the part of the book devoted to the theory of functions. Like Chapter V it is largely descriptive in character, the one being devoted to the explanation of the fundamental notions and notations concerning functions, and the other serving the same purpose for point aggregates. Attention need only be called to an interesting but much abbreviated proof that the trigonometric functions are not algebraic, to a very useful distinction between never decreasing (increasing) or monotone functions and always increasing (decreasing) or univariant functions, and to the notations $D(a)$, $V(a)$ introduced on page 153. By $D_\rho(a)$ is meant the neighborhood $a - \rho \leqq x \leqq a + \rho$ of the point a, and when only the points of a particular aggregate $\mathfrak{A}$ are under consideration, $V(a)$ stands for the points of $\mathfrak{A}$ which are in $D(a)$. The symbols $D^*(a)$ and $V^*(a)$ have the same meanings except that in each case the point a itself is excluded. The definition of the limit of a function may now be stated: the function $f(x)$ defined over an aggregate $\mathfrak{A}$ is said to have a limit η as x approaches a, if for any $D(\eta)$ there exists a $V^*(a)$ with the property that whenever x is in $V^*(a)$ the value of $f(x)$ is in $D(\eta)$. It is evident that this definition holds also when η or a is infinite, provided that proper meanings are assigned to $D(+\infty)$ and $V(+\infty)$. One needs only to notice the places in Professor Pierpont's book where many special cases of theorems on limits have been or might have been collected in a single statement and proved with a single proof by means of these notations, in order to understand how really serviceable they may be. At the end of the chapter, Cantor's example of an aggregate which is perfect in an interval $\mathfrak{A}$ but does not contain all the points of $\mathfrak{A}$, is given.

So far no mention has been made in the book of limits of functions. The fundamental definition given first in Chapter VI is what may be called the sequence definition. A function $f(x)$ is said to have the limit η as x approaches a if the values

$f(a_n)$ $(n = 1, 2, \cdots)$ have the limit η for every sequence a_n such that $a_n \neq a$ and $\lim a_n = a$. This definition has the advantage that it can be applied when η or a is infinite, and the other well-known criteria for the existence of a limit given in the succeeding sections are readily derived from it. It is to be regretted that the author has not made more use of the notations $D(a)$, $V(a)$ described above. Much space would have been saved here (§ 280–283) and in later proofs, as for example that of § 324, by grouping all the special cases involving infinites into a general one. The sections explaining the geometric interpretations of the criteria for limits are very helpful. The discussion of iterated limits is brief, only one criterion being given, although the later chapters contain many instances of iterated limits of various kinds for which special criteria have been derived in each case. Perhaps one reason why these have not all been treated by similar methods is that the criterion given by Professor Pierpont demands the existence of the double limit

$$\mathop{\mathrm{L}}_{\substack{x=a\\y=b}} f(x, y),$$

which for example in the theorem

$$\int_a^b \frac{\partial f(xy)}{\partial y}\, dx = \frac{\partial}{\partial y} \int_a^b f(xy)\, dx$$

is not of primary interest. In this connection attention may well be called to a theorem of Professor E. H. Moore * to the effect that the two iterated limits

$$\mathop{\mathrm{L}}_{y=b} \mathop{\mathrm{L}}_{x=a} f(x, y) \text{ and } \mathop{\mathrm{L}}_{x=a} \mathop{\mathrm{L}}_{y=b} f(x, y)$$

exist and are equal, provided that the two inner limits exist, and that one of them, say $\mathop{\mathrm{L}}_{x=a} f(x, y)$, is approached uniformlyin the neighborhood of $y = b$.

On pages 202–205, after a section on uniform convergence, some curious and interesting analytical representations of special functions are given. For example, the function of x whose

* Lectures on advanced integral calculus, The University of Chicago, autumn quarter, 1900; unpublished. The theorem was also stated at the Colloquium of the American Mathematical Society, held in New Haven, Conn., Sept. 5–8, 1906.

value is $+1$ for $x>0$, -1 for $x<0$, and 0 for $x=0$, denoted by signum x, is represented in the form

$$\operatorname{sgn} x = \frac{2}{\pi} \lim_{n=\infty} \operatorname{arc\,tan} nx.$$

The often-quoted function of Dirichlet, which is equal to a for rational values of x, and to b for irrational values, has then the representation

$$y = a + (b - a) \lim_{n=\infty} \operatorname{sgn} (\sin^2 n!\, \pi x).$$

After the notion of limit of a function has been made clear, the definition of a continuous function is an easy matter. In Chapter VII properties of a continuous function, such as the fact that it takes on all values between its upper and lower limits, are derived in a very attractive way, and examples are given illustrating various kinds of discontinuties.

The title of Chapter VIII is "Differentiation." The derivative is defined as usual in the more elementary books without the use of the right and left upper and lower derivatives of Dini, although the examples given of functions not derivable at certain points would serve well to illustrate them. The rules governing the use of derivatives and differentials are derived with especial care and the weaknesses common in the calculus text-books are criticised. In the conclusion of the theorem on Taylor's formula (page 255), the word "in" in the phrase "Then for any x in $\mathfrak{A}$", should be replaced by within, according to the author's usage. For the expression on the right hand side of equation (1) is not well-defined at $x=a$ on account of the derivative $f^{(n)}(c)$ which occurs in it. In the latter part of the chapter, devoted to functions of several variables, the theorem on the reversibility of differentiation for a function $f(x, y)$ could be much improved. Under the assumption that f'_x exists on $y=b$, f'_y on $x=a$, and that one of them is approached uniformly, it follows as a corollary to the theorem of Moore mentioned above, that the second derivatives f''_{xy}, f''_{yx} exist at (a, b) and are equal. The hypotheses which Professor Pierpont makes in order to prove the theorem refer to a region about the point (a, b), as must necessarily be the case if one uses the law of the mean as he does in the proof. A similar remark applies to the theorem of page 271 in which the con-

ditions are set down for a function $f(x, y)$ to be totally differentiable, that is, to be expressible in the form

$$f(a + h, b + k) = hf'_x(a, b) + kf'_y(a, b) + ah + \beta k,$$

in the neighborhood of the point (a, b), where α and β approach zero with h and k. The proof can in fact be made without using the law of the mean on the assumptions that on $x = a$ the derivative f'_x exists, is approached uniformly, and at $y = b$ is continuous, while the derivative f'_y exists at (a, b). After the proof of Taylor's formula for functions of several variables an interesting example is given, which illustrates a case in which the expansion does not hold.

The method applied in Chapter IX to implicit functions is a method of approximation due to Goursat. It is somewhat artificial but has the advantage of demanding the existence of the derivative with respect to the dependent variable only. Usually for an equation in the form

$$F(x, u) = 0,$$

the hypothesis is made that the function F has both of its first partial derivatives continuous. Professor Pierpont's is apparently the only one of the standard books in which only the derivative with respect to u is required. Systems of equations are, however, treated under the usual assumptions.

The following chapter on indeterminate forms is an exceptionally valuable one because it has to do with a subject which is badly treated in the elementary books and neglected in the more advanced treatises. The theorems are clean cut and stated with a minimum of hypothesis. In the theorem of § 452 the neighborhood $RD(a)$ should evidently be $RD^*(a)$, and at the bottom of the page the proof might have been simplified by merely referring to the theorems on the limits of a product and sum. In one or two places several theorems could have been combined into one (*e. g.* §§ 450, 451, and §§ 452, 453) with the help of the author's notation for neighborhoods, already mentioned a number of times.

Chapter XI on maxima and minima is noteworthy for the generality of the theorems for functions of a single variable, and for the pointed way in which one is warned against errors common in discussing maxima and minima of several variables. The following theorem will serve to illustrate the first point:

"Let $f(x)$ be continuous in $D(a)$. In $D^*(a)$ let $f'(x)$ be finite or infinite, and never vanish throughout any interval of it. In $RD^*(a)$, let $f'(x)$ be positive when not zero; in $LD^*(a)$, let $f'(x)$ be negative when not zero. Then $f(x)$ has a minimum at a."

On pages 317 and 321, Professor Pierpont urges the reader to distinguish carefully between the terms maximum of a function at a point, and maximum of a function in an interval. It would be better to use an entirely different word, for example upper limit or upper bound, to designate the latter. The use of the word *extreme* for either a maximum or a minimum is a helpful simplification in many places. The theorem of § 476 on extremes of functions of several variables might have been proved as briefly by a method similar to that used for functions of one variable, in which only the existence and continuity of the derivatives of the nth order, *i. e.*, the first not vanishing identically, is assumed. The examples and criticisms of §§ 479–483 bring out clearly the necessity for care in the case of functions of several variables.

The last five chapters of the book are devoted to the theory of simple proper and improper integrals, and multiple proper integrals. The first of the five is introductory and contains the usual Riemann definition of a definite integral with criteria for integrability and a discussion of certain classes of functions which are integrable. A later part of the chapter is devoted to the theory of content of a linear point aggregate $\mathfrak{A}$ from a standpoint similar to that of Jordan, and to a generalization of the definition of a definite integral by means of which it is possible to integrate a function $f(x)$ over a point set $\mathfrak{B}$ as well as over an interval. The definitions of the "outer content" of Pierpont and of "l'aire extérieure" of Jordan are different, the former being the lower bound of all intervals containing points of $\mathfrak{A}$, while the latter is the lower bound of all intervals containing points of $\mathfrak{A}$ or its boundary. The generalized integral is in fact the ordinary integral, over an interval (a, b) containing $\mathfrak{B}$, of an auxiliary function $g(x)$ which has the same value as $f(x)$ at points of $\mathfrak{B}$ but is elsewhere zero. The last two theorems of the chapter are stated as corollaries, but it is difficult to see that they follow easily from the ones immediately preceding.

In Chapter XIII further properties of proper integrals are developed. Among them one finds the two theorems of the mean, a proof of the fundamental theorem of integral calculus, and theorems on the change of variable in proper integrals.

The proof of the fundamental theorem on page 370 is one of the gems of the book, and it is to be hoped that the criticism immediately following will be widely read. The theorem of § 543 which states that the upper, and similarly the lower integrals of $f(x)$ and $f(\psi(u))\psi'(u)$, taken over corresponding intervals, are equal when $x = \psi(u)$ has a non-vanishing continuous derivative, is the key to the transformation theorems. From it the preceding theorem of § 540 follows as a corollary, and it is evident also that the part of the hypothesis in this theorem which assumes the integrability of $f(\psi(u))\psi'(u)$ is unnecessary. In fact it can be shown that if the functions $f(x)$ and $f(\psi(u))\psi'(u)$ are both integrable, one needs only to assume $\psi'(u) \geqq 0$ (or $\leqq 0$) instead of the stronger assumption $\psi'(u) \neq 0$. The theorem of § 540 applying to the case when $\psi'(u)$ is continuous but not otherwise restricted is notable for its generality and the elegance of the proof. The properties of integrals depending upon a parameter, including continuity, derivability, and integrability with respect to the parameter, are discussed in the concluding pages of the chapter. On page 387, Professor Pierpont has defined a function $f(x, y)$ to be regular in a rectangle $a \leqq x \leqq b$, $\alpha \leqq y \leqq \beta$, if for each particular value of y it is a limited integrable function of x in the interval $a \leqq x \leqq b$. It does not follow that $f(x, y)$ is limited in the whole rectangle, although for several of the proofs, notably those on pages 391, 392, such an assumption is necessary. On page 424, where the use of the word regular is still further restricted, the fact that $f(x, y)$ is limited follows from 1°. It is readily understood that the theorems mentioned above, concerning integrals involving a parameter, are really theorems on the inversion of iterated limits in which one or both of the limiting processes is an integral. Inversion is permissible in these cases under precisely the same conditions as for ordinary limits. For example in the equation

$$\lim_{y=\eta} \int_a^b f(x, y)dx = \int_a^b \lim_{y=\eta} f(x, y)dx,$$

the two outer limits exist and are equal if the two inner limits exist, and if one of them is approached uniformly. It is evident then that the hypothesis,

$$\lim_{y=\eta} \int f(x, y)dx$$

integrable, made in § 565 is superfluous. Similar alterations could be made in § 567 and § 569 where the derivative and integral with respect to y of

$$\int_a^b f(x, y)dx$$

are discussed. The inversion of integration is proved in the present chapter only for continuous functions, but in much more general form in the last chapter of the book.

The improper integrals considered in Chapters XIV and XV are those in which the integrand has a finite number of infinities, and those in which the interval of integration is infinite. The value of this part of the book lies chiefly in the thoroughness of the extension, to these two kinds of improper integrals, of the theorems well-known in the theory of proper integrals. Aside from the criteria for convergence, the properties developed are for the most part analogous to those given in the preceding chapter. At the end of Chapter XV applications are made to the B and Γ-functions. Naturally the hypotheses which must here be made in order to prove theorems on integrals involving a parameter are more complicated than for the simple integrals, and the question whether or not simplifications like those mentioned in connection with the last chapter are possible, could be answered only after careful investigation.

The final chapter of the book is on multiple proper integrals and contains properties analogous to those given in the earlier chapters for the simple integrals. As before there is a section on point aggregates, in which the properties of n-dimensional aggregates useful in connection with the multiple integrals are developed. The theory of the content of an aggregate $\mathfrak{A}$ is made to depend upon the fundamental properties of multiple integrals by means of an auxiliary function $f(x)$, which is equal to 1 at all points of $\mathfrak{A}$ and vanishes elsewhere. The upper and lower integrals of $f(x)$ in a field containing $\mathfrak{A}$ are the upper and lower contents of $\mathfrak{A}$ respectively, and when they coincide $\mathfrak{A}$ is said to be measurable. In § 722 it is shown that the notions of upper and lower multiple integrals may be generalized so that these terms have meaning when the field of integration is any limited point aggregate $\mathfrak{A}$. Furthermore the subdivisions of $\mathfrak{A}$ used in the definition need not be by planes parallel to the axes, but may be of the much more general character specified

on page 521. The treatment of multiple integrals is somewhat more general than that of simple integrals in that many of the theorems relate to the upper and lower integrals without any assumption that the function involved is integrable. One of the principal problems of the chapter is the reduction of multiple integrals to iterated integrals. The two theorems of § 733 on iterated integrals are very general, and include as a corollary a similar theorem of Pringsheim* applying to an integrable function. One of the conclusions may be stated to give an idea of the nature of the theorems:

$$\underline{\int}_{\mathfrak{A}} f d\mathfrak{A} \leqq \int_{\mathfrak{x}_i} d\mathfrak{x}_i \underline{\int}_{\mathfrak{P}_i} f d\mathfrak{P}_i \leqq \bar{\int}_{\mathfrak{x}_i} d\mathfrak{x}_i \bar{\int}_{\mathfrak{P}_i} f d\mathfrak{P}_i \leqq \bar{\int}_{\mathfrak{A}} f d\mathfrak{A}.$$

Here the dashes distinguish upper and lower integrals; $\mathfrak{x}_i$ is the projection of the aggregate $\mathfrak{A}$ on the x_i-axis and $\mathfrak{P}_i$ is the variable section of $\mathfrak{A}$ by the plane $x_i = \text{const.}$, depending therefore upon x_i. The final conclusion with regard to the reversibility of integration of a function $f(x, y)$ in a rectangle, including all the cases considered in previous chapters, is that it is possible when f is a limited function whose discontinuities on any line parallel to one of the axes, and also as a whole, form a discrete aggregate. The chapter closes with theorems on the transformation of multiple integrals of any order. Hitherto these theorems seem to have been given in detail only for double integrals.†

G. A. BLISS.

PRINCETON,
October, 1906.

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* Encyklopädie der mathematischen Wissenschaften, II A 2, p. 105.

† See Pierpont, "On multiple integrals," *Transactions Amer. Math. Society*, vol. 6 (1905), p. 416.

THE MATHEMATICAL TRIPOS OF 1906.*

Cambridge University Examination Papers, Easter term, 1906; Containing the Papers for the Mathematical Tripos, Parts I and II. Cambridge, at the University Press. Pp. 739–782. Price 2s. 6d. (For sale through Deighton, Bell and Company, Cambridge, England.)

THE main difference between the average British student of mathematics and his continental brother is summed up in the relative importance of examination and dissertation. While the examination for the German doctorate is a single oral ordeal lasting not over a couple of hours, the tripos for the first degree (baccalaureus artium) occupies several days, and the honor papers make demands on the successful candidates which could seldom be met by a young German doctor. On the other hand, the German student has been engaged on a single problem for one to two years, and his results make a more or less important contribution to the science. The student at Cambridge has concentrated all his skill on short, definite exercises, most of which, however, have a direct bearing on general mathematical development. The principal feature of the examinations is the solution of a large number of strictly original problems.

The names which have become famous among English mathematicians are almost identical with those which have stood high in the examinations, but many of these very men are now active in the movement to change their character by allowing time for the second part, and putting more emphasis on real original work, rather than on "an excessive amount of polishing of mathematical tools." †

The tripos papers of 1906 contain twenty examinations of three hours each. The number of questions varies from four to twenty, and these range in difficulty from a simple corollary in plane geometry to abstruse theorems on the outer boundary of mathematical knowledge. The papers are arranged in three groups. The first group of seven is set for all candidates for the B.A. degree with honors in mathematics. They are taken

* For an account of the development of the mathematical tripos, one should consult the History of the study of mathematics at Cambridge, by W. W. Rouse Ball, Cambridge, 1899, and the inaugural address before the London mathematical society by Dr. Glaisher, in 1888.

† See BULLETIN, volume 12, page 468.

at the end of the third year of residence, but in exceptional cases may be taken at the end of the second year. The subject matter of each examination is announced in advance; it may include a number of divisions. The questions usually consist of two parts, the first being a general theorem, frequently found in the text books, while the second part is a problem, not generally known, and probably having only a distant connection with the first part. The topics of these examinations are: (1) Euclid and conics, elementary modern geometry; (2) algebra and trigonometry; (3) analytic geometry and elementary calculus; (4) statics and dynamics; (5) geometric optics and elementary astronomy; (6) hydrostatics, heat and electricity; (7) problem paper on subjects (1) to (6). It will thus be observed that pure and applied mathematics are given equal weight. In this first part quickness and dexterity are essential to success.

After the first set of seven papers there is an interval of eleven days, at the end of which the list of honor men is published. These then take the second set of seven papers. On the result of the whole fourteen papers the candidates are arranged in order of merit as wranglers, senior optime, and junior optime. This second set of seven examinations may be roughly classified as follows: (8) analytic statics, dynamics of a particle, rigid dynamics, astronomy, hydrostatics, electricity, hydrodynamics; (9) advanced algebra (series, theory of numbers, theory of equations), symbolic operators, advanced analytics of two and of three dimensions; (10) analytic and spherical trigonometry, differential equations, definite integrals, plane curves; (11) analytic statics, dynamics, elasticity, hydrodynamics, electricity and magnetism; (12) theory of functions, elliptic functions, harmonic analysis (Bessel's functions, Fourier's functions, etc.); (13) a mixed paper, both pure and applied; (14) problem paper on (8) to (13). Here again, pure and applied mathematics are given equal weight.

About a third of the wranglers take the second part of the tripos at the end of their fourth year. In the first part a student is required to have a good general knowledge of the whole field of pure and applied mathematics, but in the second part there are four divisions of pure mathematics and four divisions of applied mathematics. The student chooses any two of these eight divisions for further specification. The questions embody the latest work in each division, and the student's rank is deter-

mined by the quality of his work in one direction. Sample papers are here added in illustration of the general character of the examination.*

PART I. SIXTH PAPER.

1. Find the conditions of equilibrium satisfied by a body which is wholly or partly immersed in a fluid, and free to turn about a fixed point.

A semi-circular lamina has one of the ends of its diameter smoothly hinged to a fixed point above the surface of a liquid, and floats with its plane vertical and its diameter half immersed. If the inclination of the diameter to the horizon is $\pi/4$, prove that the ratio of the density of the liquid to that of the lamina is $4(3\pi - 4) : 9\pi - 8$.

2. Find the center of pressure of a rectangle, completely immersed in a liquid, with one of its sides horizontal.

A rectangular block whose edges are of lengths $2a$, $2b$, $2c$ is divided by a plane through the center perpendicular to the edges of length $2c$, and the two halves are hinged together along edges parallel to those of length $2a$. The whole is then immersed in a liquid with the line of hinges inclined at an angle θ to the horizon and the dividing plane vertical, the hinges being in the upper face. Prove that the two halves will not separate unless

$$\left\{\left(1 - \frac{\sigma}{\rho}\right)c^2 - \tfrac{2}{3}b^2\right\}\cos\theta > 2bd,$$

where d is the depth of the center of gravity of the block, σ is the density of the block, and ρ that of the liquid.

3. A cylindrical diving-bell is lowered to a given depth below the surface of a liquid. Find the height to which the liquid rises inside the bell.

A cube of density ρ_0 floats on a liquid of density ρ, and a cylindrical diving-bell, of length l and cross section A, is placed over it and lowered until the cube just touches the top. Show that the height to which the liquid rises inside the bell is

$$[A(\rho - \sigma) - \rho a^2]\,[l(\rho - \sigma) - a(\rho - \rho_0)]\,/\,[\rho(\rho - \sigma)(A - a^2)],$$

* We are permitted to reprint the questions by the courtesy of the Syndics of the Cambridge University Press, who hold the copyright. Complete sets of the tripos can be obtained from the publishers.

where σ is the density of the atmosphere, and a is the length of an edge of the cube.

4. Describe Smeaton's air-pump, and find the density in the receiver after n strokes.

A barometer tube, in the form of a flat-topped cylinder, is placed with its lower end in a vessel containing mercury, and the air is pumped out. Prove that, if x_n is the distance of the mercury from the top of the tube after n strokes,

$$(x_n - h)(ax_n + u) = (x_{n-1} - h)ax_{n-1},$$

where a is the cross section of the tube, u the volume of the barrel of the pump, and h the ultimate value of x_n.

5. Define the terms 'latent heat' and 'specific heat,' and show how the specific heat of a substance may be determined by observing the alteration in temperature of a given mass of water, into which a given mass of the substance at a given temperature is dropped.

A closed vessel of mass 40 grams and specific heat .095 is filled with 4.4 grams of a gas, the whole being at temperature 20° C. The vessel is then enclosed in a chamber through which steam at 100° C. is passed until the temperature becomes uniform, and the increase in mass is found to be .68 gram. Prove that the specific heat of the gas under the given conditions is .173, the latent heat of steam at 100° C. being 536.5.

6. Define *potential*, and find the force and the potential at any point due to a freely electrified spherical conductor.

A curve is drawn on the surface of a freely electrified oval conductor surrounded by air, dividing it into two parts on which the charges are E and E' respectively. The space, enclosed by the portion of the surface of the conductor on which the charge is E' and by the lines of force drawn from every point of the curve to infinity, is filled with a dielectric of S. I. C. K. Prove that the potential of the conductor is now diminished in the ratio $E + E' : E + KE'$.

7. Prove that the energy of a system of charged conductors is $\frac{1}{2}\Sigma EV$.

A condenser is formed of two parallel plates. Prove that, when the potentials of the plates are kept constant, the work done by the system in a small displacement is equal to the increase in the energy of the system.

8. Define the coefficients of potential of a system of conductors, and prove that $p_{rs} = p_{sr}$. Under what circumstances does $p_{rr} = p_{rs}$?

Three conductors A_1, A_2 and A_3 are such that A_3 is practically inside A_2. A_1 is alternately connected with A_2 and A_3 by means of a fine wire, the first contact being with A_3. A_1 has a charge E initially, A_2 and A_3 being uncharged. Prove that the charge on A_1, after it has been connected with A_3 n times, is

$$\frac{E\beta}{\alpha+\beta}\left\{1+\frac{\alpha(\gamma-\beta)}{\beta(\alpha+\gamma)}\left(\frac{\alpha+\beta}{\alpha+\gamma}\right)^{n-1}\right\},$$

where α, β and γ stand for $p_{11}-p_{12}$, $p_{22}-p_{12}$ and $p_{33}-p_{12}$ respectively.

9. State and prove Kirchhoff's laws as to the distribution of electric currents in a system of linear conductors containing given electromotive forces.

A square $ABCD$ is formed of a uniform piece of wire, and the center is joined to the middle points of the sides by straight wires of the same material and cross section. A current is taken in at A and is drawn off at the middle point of BC. Prove that the equivalent resistance of the network between these points is $\frac{29}{48}$ of that of a side of the square.

10. Describe the tangent galvanometer.

A given current sent through the galvanometer deflects the magnet through an angle θ. The plane of the coil is slowly rotated round the vertical axis through the center of the magnet. Prove that, if $\theta > \frac{1}{4}\pi$, the magnet will describe complete revolutions, but if $\theta < \frac{1}{4}\pi$, the magnet will oscillate through an angle $\sin^{-1}(\tan\theta)$ on each side of the meridian.

PART I. TENTH PAPER.

1. If $\sum_0^\infty u_n$ is a semi-convergent series of real terms, prove that the infinite product $\prod_0^\infty (1+u_n)$ tends towards a positive limit or towards zero according as $\sum_0^\infty u_n^2$ is convergent or divergent.

If $u_0 = u_1 = u_2 = 0$ and when $n > 1$

$$u_{2n-1} = \frac{-1}{\sqrt{n}}, \qquad u_{2n} = \frac{1}{\sqrt{n}} + \frac{1}{n} + \frac{1}{n\sqrt{n}},$$

show that $\sum_0^\infty u_n$ and $\sum_0^\infty u_n^2$ are both divergent, but that $\prod_0^\infty (1+u_n)$ is convergent.

2. Show that a rational integral congruence of degree n cannot have more than n roots incongruent to a prime modulus.

Prove that, if n is an odd prime,

$$x^{n-1} - 1 - [x^2 + 1(n-1)]\,[x^2 + 2(n-2)] \cdots [x^2 + \tfrac{1}{2}(n-1)\tfrac{1}{2}(n+1)] \equiv 0 \pmod{n}$$

is an identical congruence.

What arithmetic theorems result from this?

3. Investigate the solution of the quartic $ax^4 + 4bx^3 + 6cx^2 + 4dx + e = 0$, and obtain the auxiliary cubic in the form $4a^3\theta^3 - Ia\theta + J = 0$.

If ϕ is the cross-ratio of the four roots, show that the correspondence between θ and ϕ may be written in the forms

$$(\phi+1)^2 I = 12a^2\theta^2(\phi^2-\phi+1), \quad (\phi+1)^2 J = 4a^3\theta^3(2\phi-1)(\phi-2).$$

4. Show that the hessian of a quantic is a covariant of that quantic.

If a ternary cubic is the product of three linear factors, prove that its hessian is the product of the cubic with a constant.

5. If u is a homogeneous function of $x_0, x_1, x_2, \cdots, x_m$ of degree n prove that

$$\left(x_0\frac{\partial}{\partial x_0} + x_1\frac{\partial}{\partial x_1} + \cdots + x_m\frac{\partial}{\partial x_m}\right)u = nu.$$

Let

$$\Omega \equiv x_0\frac{\partial}{\partial x_1} + 2x_1\frac{\partial}{\partial x_2} + 3x_2\frac{\partial}{\partial x_3} + \cdots + mx_{m-1}\frac{\partial}{\partial x_m},$$

and

$$O \equiv mx_1\frac{\partial}{\partial x_0} + (m-1)x_2\frac{\partial}{\partial x_1} + (m-2)x_3\frac{\partial}{\partial x_2} + \cdots + x_m\frac{\partial}{\partial x_{m-1}},$$

and let G be a homogeneous algebraic function of $x_0, x_1, \cdots, x_m$ of degree n such that the sum of the suffixes in each term is w.

Prove that $(\Omega O - O\Omega)G = (mn - 2w)G$.

6. If x, y, z are connected by a single equation, express

$$\frac{\partial z}{\partial x}, \quad \frac{\partial z}{\partial y}, \quad \frac{\partial^2 z}{\partial x^2}, \quad \frac{\partial^2 z}{\partial x \partial y}, \quad \text{and} \quad \frac{\partial^2 z}{\partial y^2}$$

in terms of differential coefficients of x with respect to y and z.

If

$$\frac{\partial(Z, z)}{\partial(x, y)} = 0, \quad \text{where} \quad \frac{\partial^2 z}{\partial x \partial y} - Z \frac{\partial z}{\partial x} \frac{\partial z}{\partial y} = 0,$$

show that the same relations hold good on permuting the letters, that is to say

$$\frac{\partial(X, x)}{\partial(y, z)} = 0, \quad \text{where} \quad \frac{\partial^2 x}{\partial y \partial z} - X \frac{\partial x}{\partial y} \frac{\partial x}{\partial z} - 0, \quad \text{etc.}$$

7. Find the equation of a pair of tangents to $x^2/a^2 + y^2/b^2 = 1$, the chord of contact being $\lambda x + \mu y + \nu = 0$.

Find the equation of a circle touching these two tangents and meeting the ellipse in the chord $x \cos \phi + y \sin \phi = p$, concurrent with the chords of contact; and if ϕ is the constant a, show that the locus of the intersection of the above-mentioned tangents is the conic

$$x^2/(a^2 \cos^2 \alpha) - y^2/(b^2 \sin^2 a) = (a^2 - b^2)/(a^2 \cos^2 a + b^2 \sin^2 a),$$

that is to say, an hyperbola confocal with a given ellipse.

8. Find the point equation of the circular points at infinity in any system of homogeneous coördinates, and deduce the corresponding tangential equation.

A parabola circumscribes a triangle, show that its axis touches a curve of the third class (a three-cusped hypocycloid) given by

$$\lambda^2 \frac{\partial \Omega}{\partial \lambda} + \mu^2 \frac{\partial \Omega}{\partial \mu} + \nu^2 \frac{\partial \Omega}{\partial \nu} = 0,$$

and the tangent at the vertex always touches the curve of the sixth class defined by the rationalized form of

$$\lambda^{\frac{1}{2}} \frac{\partial \Omega}{\partial \lambda} + \mu^{\frac{1}{2}} \frac{\partial \Omega}{\partial \mu} + \nu^{\frac{1}{2}} \frac{\partial \Omega}{\partial \nu} = 0,$$

where the line at infinity is $x + y + z = 0$ in point-coördinates, and the rational tangential equation of the circular points at infinity is $\Omega = 0$.

9. Find the equation of a normal at any point of the ellipsoid

$$x^2/a^2 + y^2/b^2 + z^2/c^2 = 1,$$

and deduce that normals meeting in (x_0, y_0, z_0) lie on the cone

$$\frac{x_0}{x - x_0}(b^2 - c^2) + \frac{y_0}{y - y_0}(c^2 - a^2) + \frac{z_0}{z - z_0}(a^2 - b^2) = 0.$$

If (x_0, y_0, z_0) lies on the ellipsoid and tangent planes be drawn to the ellipsoid at the five feet of normals other than (x_0, y_0, z_0), then the feet of the perpendiculars from the center on these tangent planes all lie in the tangent plane at $(-x_0, -y_0, -z_0)$, and are all situated on a rectangular hyperbola passing through the foot of the perpendicular from the center on the same plane.

10. Find the principal planes of the quadric

$$S \equiv ax^2 + by^2 + cz^2 + 2fyz + 2gzx + 2hxy - 1 = 0.$$

If $X \equiv ax + hy + gz$, $X' \equiv a'x + h'y + g'z$, etc., show that the set of conjugate diametral planes which $S = 0$, $S' = 0$ have in common is determined by the equation

$$\frac{\partial(S, S', S'')}{\partial(x, y, z)} = 0,$$

where S'' is given by either $(A, B, C, F, G, H)(X', Y', Z')^2$ or $(A', B', C', F', G', H')(X, Y, Z)^2$.

Part II. Divisions II, IV.

1. Prove that covariants of binary forms can be expressed in terms of symbolical factors of the type (ab) and a_x, and explain how to extend the result to ternary forms.

Prove that for binary forms the minimum weight of a perpetuant of degree four is seven, and explain how to extend the result to higher degrees.

2. A line l meets a conic S_1 in two conjugate imaginary points, P, Q. O is a point on a conic S_2 and OP, OQ meet S_2 in $P'Q'$; obtain a real construction for the line $P'Q'$.

One pair of opposite vertices of a quadrilateral are real and a second pair are conjugate and imaginary. Give a real construction for the third diagonal and the third pair of vertices.

State, in terms of cross ratios, the proposition that the angle at the center of a circle is double that at the circumference on the same arc, and give a direct proof of the statement.

3. Give an account of the properties of the twenty-seven

lines on a cubic surface, and explain how to deduce properties of the bitangents of a plane quartic curve.

Show that there exist sets of six bitangents which touch the same conic.

4. Prove the fundamental theorems relating to the parametric expressions for the coördinates of a point on a plane curve of deficiency 0 or 1.

In the case of a plane cubic find how many lines can be drawn such that, at each point where they meet the curve, a conic can be drawn having six-point contact with the curve.

Indicate briefly how the equation of any elliptic curve can be reduced to the form

$$y^2 = 4x^3 - g_2x - g_3.$$

VIRGIL SNYDER.

SHORTER NOTICES.

Ueber die Entwicklung der Elementar-Geometrie im XIX. Jahrhundert. Bericht der Deutschen Mathematiker-Vereinigung, erstattet von MAX SIMON. Mit 28 Figuren im Text. Leipzig, B. G. Teubner, 1906. viii + 278 pp. Price, 8 marks.

Methodik der Elementaren Arithmetik in Verbindung mit Algebraischer Analysis. Von MAX SIMON. Mit 9 Textfiguren. Leipzig, B. G. Teubner, 1906. vi + 108 pp. Price, 3 marks 20 pf.

AMONG those occupying chairs of mathematics in the German universities there is no one who takes greater interest in the work of the secondary teacher than Professor Simon. He is earnest in his advocacy of reform, zealous in his application of the history of mathematics to the principles of teaching, and full of that good-humored argument that makes a man acceptable as a speaker in an assembly of teachers. Therefore it comes about that Professor Simon is able to command appreciative audiences for his addresses and a goodly circle of readers for his numerous literary efforts. It is rather in his work as a speaker, however, that he is most successful. His fund of enthusiasm, his genial countenance, and his action in address, all tend to lead his hearers to consider his arguments as wholes, without attending to minor inaccuracies of language or of state-

ment. When, on the other hand, he attempts to put his ideas on paper, his larger thoughts, worthy though they may be, seem submerged in a sea of minor inaccuracies. In the case of the two books under review, the first is of such a nature as to invite even more errors than are usually found in Professor Simon's writings, and therefore, as may be inferred from what has been said, the reader will find much to vex his soul in looking over its pages.

In spite of these annoyances, however, each book has a distinct value, and each can be recommended to the teacher of mathematics, if not to the mere mathematician. The first book, "Ueber die Entwicklung der Elementar-Geometrie im XIX. Jahrhundert," is taken from the Jahresbericht der Deutschen Mathematiker-Vereinigung, and was originally begun in the interests of the Encyklopädie der mathematischen Wissenschaften. Its scope is limited, as the title suggests, in time to the nineteenth century, and in subject to the geometry of the Gymnasium or Oberrealschule. Professor Simon begins with a general survey of the bibliography of the subject, particularly that of the history, the method of teaching, and the text-books relating to geometry. It is in such lists of books and articles, imperfect though they are, that the chief value of the work will be found to lie. The author then takes up a number of special topics as follows: The theory of parallels; the circle, including quadrature, inscriptibility of polygons, and the multisection of arcs; areas, including isoperimetry; the triangle, including a study of critical points and lines; polygons; plane configurations, including similarity, centers of gravity, and transversals; general space relations, in particular certain questions of stereometry, volumes and elementary surfaces, and spherics; special space relations, including the tetrahedron, polyhedron, and the Euler theorem; and trigonometry, plane and spherical.

Under each of these topics a brief historical survey is given, with a bibliography, and this is followed by a summary of the most important attempts at treating the subject in hand. Thus under the quadrature of the circle the historical survey is followed by a list (rather imperfect) of circle-squarers, a statement of approximate constructions, a few of the efforts at representing π numerically, and the history of a few such attempts as that of Hippocrates. In the same way the trisection problem, the 17-gon, the critical points of the triangle, and Euler's

problem (with no reference to Hadamard's proof) are discussed. In no case is the treatment as thorough as in Klein's Vorträge, but it is always readable, and the bibliography opens up to the student a good part of the literature of the subject.

The "Methodik der elementaren Arithmetik" is a set of lectures delivered by Professor Simon in the summer semester of 1904. As such, they would naturally be expected to be more popular and less critical in details than the work already mentioned. They are simple in style, and were doubtless intended for teachers who felt the need of having their attention called to the modern treatment of certain fundamental ideas of mathematics. Hence we find a popular treatment of the idea of integral number, the operations with number and the fundamental laws involved, fractions, series, powers and roots, equations of the first three degrees, logarithms, and complex numbers. It is such a course of lectures as might profitably be attended by the high school teacher in this country, and the book should find a considerable circle of readers here as well as in Germany.

While the books have this value, however, it must be said that the array of inaccuracies which confronts the reader seems quite inexcusable. The number of the errata at the end of the Entwicklung, upwards of 135, might easily be doubled from a single reading. Moreover there are other and more serious points of criticism. Professor Simon's dismissal of Dr. H. Schotten's "Inhalt und Methode" by the simple word "schwächer" is unwarranted, for the book contains a good deal that is of value to the teacher of geometry. His statement that Rupert's little book on famous propositions of geometry seems to be merely an extract from Ball cannot be justified; if the book were to be attacked it should be on other grounds. Moreover Professor Simon's knowledge of the relative values of books, monographs, and proofs of a non-German character is limited, as the merest glance at his list of text-books will show. An example of this is seen in his inclusion of an edition of Sacrobosco's Sphæra in a list of works on modern spherics. His statement (Methodik, p. 16) that in Sargon's time the position system was known is liable to serious misconstruction. His statement that the French call the multiplication table the "table de Pythagore" will also be misunderstood, for the name was a common one in the works of the mediæval writers. That Gemma Frisius said "unitas ipsa non est numerus," in 1576

(Methodik, page 8), is impossible, for he was dead then. Moreover it was not at all a new idea when he published his arithmetic in 1540, for it had been generally repeated by writers from the time of Boethius down. His statement that the study of the Pascal triangle began with Stifel is also incorrect, for it was apparently known long before Stifel's works were written, appearing for example in the engraved title page of "Eyn Newe Vnnd wolgegründte vnderweysung aller Kauffmanss Rechnung" published by Apianus at Ingolstadt in 1527.

DAVID EUGENE SMITH.

Elements of Descriptive Geometry. By O. E. RANDALL. Boston, Ginn & Company, 1905. iv + 209 pp.

Elements of Descriptive Geometry. By CHARLES E. FERRIS. New York, American Book Company, 1905. vii + 127 pp.

THESE books meet the demand for texts in which the objects are shown in the figures in the third quadrant. They follow the method of Warren and Church in presenting first the analysis and then the construction of a given problem. The figures are not in a separate volume or collection of plates, but in the text where most convenient for the reader.

The authors present the fundamental ideas with care, and Dr. Randall supplements the figures for the first fifty pages with "pictorial drawings" showing the projection planes and the object as they actually appear to the eye of the observer.

It will be noticed that the chapters on "Lines and surfaces" retain the definitions and principles used by the older American writers. If it is necessary to set up for engineering students a complete theory of curves and surfaces without analysis, it is desirable that in doing it naïve expressions that may confuse students of mathematics should be avoided.* For example, in Dr. Randall's book, § 203, we read * * * "The portion of the line generated by the point while moving from one position to its consecutive position is called an *elementary line* and while in theory it may be regarded as having length, practically speaking it has none."

The error of classifying a warped surface as a surface of single curvature is implied by Dr. Randall in § 221, where we find:

*Compare Rohn, Lehrbuch der darstellenden Geometrie, Leipzig, 1906, p. 219.

"Depending upon the character of the generatrix we have two classes of surfaces: first those which are generated by straight lines or those which have rectilinear elements; and second those which are generated by curved lines known as *surfaces of double curvature.*"* Professor Ferris also states in Chapter IV on "Double curved surfaces" that "double curved surfaces have no right line elements" and he classes warped surfaces by themselves.

Monge's original sharp classification of surfaces into planes, single curved or *developable* surfaces, and double curved surfaces† has been mangled by several American writers including Warren‡ and Church.§ The present classification by Dr. Randall into single curved surfaces, § 222; surfaces of revolution, § 223; and double curved surfaces of revolution, § 224; seems without adequate foundation.

Professor Ferris has a brief chapter on shades and shadows and one on perspective.

L. I. HEWES.

NOTES.

THE Thirteenth Annual Meeting of the AMERICAN MATHEMATICAL SOCIETY will be held in New York on Friday and Saturday, December 28–29. Friday morning will be devoted to a joint session with Section A of the American association for the advancement of science and the Astronomical and astrophysical society of America. President OSGOOD'S address will probably be delivered at the opening of the Friday afternoon session. The Council will meet on Friday afternoon. The annual election of officers and other members of the Council will be held on Saturday morning. Special railroad rates will be available for this meeting.

THE concluding (October) number of volume 7 of the *Transactions* of the AMERICAN MATHEMATICAL SOCIETY contains the following papers: "Weierstrass' theorem and Kneser's theorem on transversals for the most general case of an extremum of a

* See also Randall, §§ 223, 224.

† G. Monge, Géométrie descriptive. Paris, 1858, p. 120.

‡ S. Edward Warren, Descriptive geometry. New York, 1860, p. 4 and p. 194.

§ A. E. Church, Descriptive geometry and shades and shadows. New York, 1870, p. 41.

simple definite integral," by O. BOLZA; "Area of curved surfaces," by J. PIERPONT; "On multiply transitive groups," by W. A. MANNING; "Zur Theorie der vollständig reduciblen Gruppen, die zu einer Gruppe linearer homogener Substitutionen gehören," by L. STICKELBERGER; "On commutative linear algebras in which division is always uniquely possible," by L. E. DICKSON; "On the order of linear homogeneous groups," by H. F. BLICHFELDT; "On automorphic groups whose coefficients are integers in a quadratic field," by J. I. HUTCHINSON; "A class of periodic solutions of the problem of three bodies with applications to the lunar theory," by F. R. MOULTON; "A problem in the reduction of hyperelliptic integrals," by J. H. MCDONALD; "On the differential invariants of a plane," by C. N. HASKINS. Notes and errata, volumes 6, 7.

Professor W. F. OSGOOD has retired from the Editorial Committee of the *Transactions*, and is succeeded by Professor H. S. WHITE.

THE opening (October) number of volume 8 of the *Annals of Mathematics* contains the following papers: "The fundamental laws of addition and multiplication in elementary algebra," by E. V. HUNTINGTON; "On a criterion of Pringsheim's for expansibility in Taylor's series," by M. B. PORTER; "Multiply perfect numbers of three different primes," by R. D. CARMICHAEL.

AT the Stuttgart meeting of the Deutsche Mathematiker-Vereinigung, Professor A. VON BRILL, of the University of Tübingen, was elected president; Professors M. KRAUSE, of Dresden, and A. SCHOENFLIES, of Königsberg, were elected to the Council. The next meeting will be held at Dresden during the summer of 1907. The society now numbers 688 active members.

THE Cambridge University Press announces that the following works will soon appear: Theory of differential equations, volume 5, by A. R. FORSYTH; Theoretic mechanics, an introductory treatise on the principles of dynamics, with applications and numerous examples, by A. E. H. LOVE.

THE following parts of the Encyklopädie der mathematischen Wissenschaften are announced by the firm of B. G. Teubner, Leipzig, to be in the press, and will probably appear in December: III, 1, Analysis situs, by P. HEEGARD and M. DEHN; V, 1, Kinetische Theorie der Materie, by L. BOLTZMANN

and J. NABL; Kapillarität, by H. MINKOWSKI; V, 2, Beziehungen zwischen electrostatischen und magnetischen Zustandsänderungen einerseits und elastischen und thermischen andererseits, by F. POCKELS. Also those mentioned in the BULLETIN, volume 12, page 465, except III, 2, Abzählende Methoden, by H. G. ZEUTHEN; and Allgemeine Theorie der höheren ebenen algebraischen Kurven, by L. BERZOLARI, which have already appeared.

Of the French edition of the encyclopedia there have thus far appeared the first sections of volumes I 1, 3 and 4.

THE following Americans have received the doctorate from European universities, with mathematics as the major subject, during the six semesters ending August 1, 1906: J. W. BRADSHAW (Strassburg, 1904), "Ueber die Flächendichtigkeit der Elektricität auf unendlich langen Cylindern;" J. L. COOLIDGE (Bonn, 1904), "The dual projective geometry of elliptic and spherical space"; F. J. DOHMEN (Greifswald, 1904), "Darstellung der Berührungstransformationen in Konnexcoordinaten"; D. C. GILLESPIE (Göttingen, 1906), "Anwendungen des Unabhängigkeitsatzes auf die Lösung der Differentialgleichung der Variationsrechnung"; A. L. P. WERNICKE (Göttingen, 1904), "Ueber die Analysis Situs mehrdimensionaler Räume"; W. D. A. WESTFALL (Göttingen, 1905), "Zur Theorie der Integralgleichungen."

To the list of American doctorates for 1906 having mathematics for the major subject should be added the names of W. R. LONGLEY (Chicago), "A class of periodic orbits of an infinitesimal body subject to the attraction of n finite bodies;" and F. L. GRIFFIN (Chicago), "Certain periodic orbits of k finite bodies revolving about a relatively large central mass."

UNIVERSITY OF PARIS.—The following courses in mathematics are given during the present semester:—By Professor G. DARBOUX: Systems of curvilinear coördinates, two hours.—By Professor E. GOURSAT: Elements of the theory of analytic functions, two hours; Conference, one hour.—By Professor P. PAINLEVÉ: General laws of motion and equilibrium, two hours.—By Professor L. RAFFY: Applications of analysis to geometry, one hour; General mathematics, two hours.—By Professor H. POINCARÉ: Limitations of the newtonian law, two hours.—By Professor J. BOUSSINESQ: Cooling of

liquids, two hours. — By Professor G. KOENIGS: Dynamics of machines, two hours. — By Professor E. BOREL: Theory of functions, one hour. — By Professor J. HADAMARD: General mathematics, two hours; Conference, four hours. — By Dr. P. PUISEUX: Conference on mechanics, two hours. — By Dr. E. BLUTEL: Conference on algebra, two hours. In the École Normale. — By Professor J. TANNERY: Calculus, two hours, and the courses mentioned above, by Professors Hadamard, Raffy and Borel.

PROFESSOR J. MOLK, of the University of Nancy, and Professor T. LEVI-CIVITA, of the University of Padua, have been elected to membersip in academy of sciences at Halle.

PROFESSOR G. ROST, of the University of Würzburg, has been promoted to a full professorship of mathematics.

PROFESSOR W. O. WIEN, of the University of Würzburg, has declined the invitation of the University of Berlin to be professor of theoretical physics, as successor to the late Professor Drude.

DR. M. BOHREN has been appointed docent in mathematics and theory of probabilities at the University of Bern.

PROFESSOR G. LANDSBERG, of the University of Breslau, has been appointed associate professor of mathematics at the University of Kiel.

PROFESSOR TH. SCHMID, of the technical school at Vienna, has been promoted to a full professorship of descriptive geometry.

PROFESSOR K. ZSIGMONDY, of the technical school at Prague, has been appointed professor of mathematics at the technical school of Vienna.

DR. H. REISSNER, of the technical school at Charlottenburg, has been appointed professor of mathematics at the technical school at Aachen.

AT the University College of London, Mr. N. G. DUNBAR has been appointed demonstrator in applied mathematics.

At the University of Michigan, Dr. J. W. GLOVER has been promoted to a junior professorship of mathematics and insurance. The insurance courses have been broadened by the addition of a seminary for advanced students.

AT Lehigh University, Mr. J. H. OGBURN has been appointed assistant professor of mathematics.

AT George Washington University, Mr. P. N. PECK has been promoted to an assistant professorship of mathematics, and Mr. G. A. Ross has been appointed instructor in mathematics.

AT the College of the City of New York, Mr. S. J. MAGARGE has been appointed tutor in mathematics. Mr. R. F. DEIMEL has been transferred to the department of physics.

THE following academic appointments are also announced: DR. F. L. GRIFFIN as instructor in mathematics at Williams College, Williamstown, Mass.; Mr. R. M. BARTON and Mr. F. C. MOORE as instructors in mathematics at Dartmouth College; Mr. P. DORWEILAR and Mr. C. F. HAGENOW as instructors in mathematics at the Armour Institute of Technology; Mr. F. F. DECKER as instructor in mathematics at Syracuse University; Mr. L. WEEKS as instructor in mathematics at Purdue University; Mr. W. M. SMITH as tutor and Mr. J. W. COLLITON as instructor in mathematics at Lafayette College; Mr. W. S. PEMBERTON and Mr. P. F. HEDGES as assistants in mathematics at the University of Missouri.

PROFESSOR ERNESTO CESÀRO, of the University of Naples, died September 12, at the age of forty-seven years.

CATALOGUES of second-hand mathematical books: C. Winter, Dresden, Galleriestrasse 8, catalogue no. 97, 287 titles in pure and applied mathematics. — H. Sotheran and Co., London, 140 Strand, catalogue 666, 1437 titles in mathematics and sciences. — Carlo Clausen, Turin, Via Po 11, catalogue no. 128, 678 titles in mathematics, physics, and astronomy. — Karl W. Hiersemann, Leipzig, Königstrasse 3, Katalog 304, a few 16th–18th century works.

NEW PUBLICATIONS.

I. HIGHER MATHEMATICS.

BIBLIOGRAPHY of Russian literature upon mathematics, pure and applied, and upon the natural sciences, for 1902. Edited by the Society of naturalists of Kiev. Kiev, 1905. 8vo. (Russian.)

BORELIUS (J.). Zur Theorie der Filarevoluten und Parallelkurven ebener und sphärischer Kurven. Lund, 1906. 8vo. 63 pp. M. 2.00

DEPÈNE. Ueber die Flächen gewisser einem Dreieck eingeschriebener Dreiecke. Breslau, 1906. 4to. 13 pp. M. 1.50

FATTOR (L.). Di un metodo per determinare le regole di divisibilità relative ad un numero primo con 10. Cividale, Stagni, 1906. 8vo. 7 pp.

——. Una regola pratica per avere il periodo di $1/p$ essendo p primo con 10. Cividale, Stagni, 1906. 8vo. 4 pp.

GALILEI (G.). Le opere. Edizione nazionale sotto gli auspici di Sua Maestà il Re d'Italia. Vol. XVIII. Firenze, Alfani, 1906. 4to. 545 pp.

GEPP (H.). Ueber Inversionssummen. (Diss.) Giessen, 1906. 8vo. 40 pp.

HAWLITSCHEK (K.). Integration irrationaler Differentiale. (Progr.) Prag-Neustadt, 1906. 8vo. 9 pp.

HEYDEN (A. F. V.). First steps in the calculus. London, Arnold, 1906. 8vo. 224 pp. Cloth. 3s.

JAHRBUCH über die Fortschritte der Mathematik. Begründet von C. Ohrtmann. Unter Mitwirkung von F. Müller und A. Wangerin sowie der Berliner Mathematischen Gesellschaft herausgegeben von E. Lampe. Vol. 35. Heft 1. Berlin, 1906. 8vo. 496 pp. M. 17.80

KOERBER (F.). Transformator für sphärische Coordinaten. Berlin, Reimer, 1906. 8vo. M. 1.50

LANNER (A.). Die neueren Darstellungen der Grundprobleme der reinen Mathematik. Berlin, Salle, 1906. 8vo. M. 3.00

LEBESGUE (H.). Leçons sur les séries trigonométriques. Paris, Gauthier-Villars, 1906. 8vo. 7 + 128 pp. F. 3.50

MCCORMACK (T. J.). See MACH (E.).

MACH (E.). Space and geometry in the light of physiological, psychological and physical inquiry; from the German by T. J. McCormack. Chicago, Open Court, 1906. 12mo. 2 + 148 pp. Cloth. $1.00

MANSION (P.). Précis de la théorie des fonctions hyperboliques, suivi d'une théorie purement analytique des fonctions circulaires. 2e édition, revue et augmentée. Paris, Gauthier-Villars, 1906. 8vo. 44 pp. F. 1.00

MEYER (P.). Beweis eines von Euler entdeckten Satzes, betreffend die Bestimmung von Primzahlen. Strassburg, 1905. 8vo. 29 pp. M. 1.50

PETIT BOIS (G.). Tables d'intégrales indéfinies. Liége, 1906. 4to. 12 + 154 pp. M. 8.50

PIEL (C.). Ueber die Kegelschnitte, welche durch 3 Punkte und 2 Tangenten oder durch 2 Punkte und 3 Tangenten bestimmt sind, und die Kegelschnittsysteme ($3\,p$, $1\,l$) und ($1\,p$, $3\,l$). Strassburg, 1905. 8vo. 73 pp. M. 2.00

POINCARÉ (H.). El valor de la ciencia. Versión española de Emilio González Llana. Madrid, Gutenberg, 1906. 264 pp.

PUCCIANO (G.). Indagine sul potenziale logaritmico e newtoniano. Messina, d'Amico, 1906. 8vo. 38 pp.

RANTSOV. Pure mathematics. With a popular exposition of the foundations of higher analysis. St. Petersburg, 1906. 8vo. (Russian.) M. 3.60

ROITMAN (D.). Mathematics as a science and as a subject of general culture. St. Petersburg, 1906. 8vo. 56 pp. (Russian.) R. 0.50

STEINGRÄBER (W.). Ueber partielle Differentialgleichungen erster Ordnung im R_4. Greifswald, 1906. 8vo. 46 pp. M. 1.50

TOGNELLI (G.). Una applicazione della teoria dei residui di variabile complessa. Livorno, Belforte, 1906. 8vo. 49 pp.

II. ELEMENTARY MATHEMATICS.

BIELER (A.). Arithmetisches Lehr- und Uebungsbuch für Knaben-Mittelschulen. Teil I: Bis zu den Gleichungen zweiten Grades mit mehreren Unbekannten einschliesslich. Teil II: Reihenlehre, Zinseszins-Rechnung und Anfangsgründe der Trigonometrie. Leipzig, Teubner, 1906. 8vo. 323 pp. Cloth. M. 2.00

CERAMICOLA (F.). Multipli, summultipli, rapporti e proporzioni di grandezze omogenee commensurabili. Arezzo, Sinatti, 1906. 8vo. 33 pp.

CHILD (J. M.). See LOCK (J. B.).

CRACKNELL (A. G.). See WORKMAN (W. P.).

DEAKIN (R.). New matriculation algebra. With a selection of graphs. 4th edition. London, Clive, 1906. 8vo. 512 pp. Cloth. (University tutorial series.) 3s. 6d.

EDWARDES (E. J.). Elements of plane geometry. London, Arnold, 1906. 8vo. 226 pp. Cloth. 3s. 6d.

F. I. C. Eléments d'algèbre, avec de nombreux exercices. Paris, Poussielgue, 1906. 16mo. 8 + 413 pp.

FEIST (A.). Mathematische Aufgaben. Verzeichnis der mathematischen Aufgaben, die bei den Reifeprüfungen der 9-stufigen höheren Lehranstalten des Herzogtums Braunschweig von Ostern 1892 bis Ostern 1903 bearbeitet worden sind. Braunschweig, 1906. 4to. 35 pp. M. 1.50

GÉOMÉTRIE. Cours supérieur; par une réunion de professeurs. Paris, Poussielgue, 1906. 16mo. 8 + 323 pp.

HELLGREN (A. E.). Allmänna metoder vid plangeometriska teorems och problems lösning. Part 1: 3d edition. Part II: 2d edition. Stockholm, 1906. 8vo. 239 pp. Cloth. M. 7.00

HOPFNER (F.). Beitrag zur konstruktiven Lösung planimetrischer Aufgaben. (Progr.) Prag-Neustadt, 1906. 8vo. 9 pp.

HOYOS Y JULIÁ. Nociones y ejercicios de aritmética y geometria. Zamora, 1906. 159 pp.

LOCK (J. B.) and CHILD (J. M.). Trigonometry for beginners. London, Macmillan, 1906. 8vo. 204 pp. Cloth. 2s. 6d.

MAISON (C. C.). Trigonometry simplified. A practical treatise on the subject of trigonometry. Cleveland, 1906. 12mo. 61 pp. $0.50

MALONE (F.). Mathematical dexterities; or, the shortest way possible to reach accurate and quick results: a self-instructor for home study and a guide and reference to all who desire to be practical in business life. St. Louis, Malone, 1906. 16mo. 164 pp. Cloth. $1.00

MARTUS (H. C. E.). Mathematische Aufgaben zum Gebrauch in den obersten Klassen höherer Lehranstalten. Aus den bei Reifeprüfungen an den deutschen höheren Schulen gestellten Aufgaben ausgewählt. Teil IV: Ergebnisse des Teils III. 2te, vermehrte Auflage. Dresden, 1906. 8vo. Cloth. M. 5.60

MATRICULATION model answers: Mathematics. Being the London university matriculation papers in mathematics from September, 1902, to June, 1906. London, Clive, 1906. 8vo. Cloth. (University tutorial series.) 2s.

MILLIS (J. F.). See STONE (J. C.).

MINET (A.) et PATIN (L.). Cours pratique d'arithmétique, de système métrique et de géométrie (Cours élémentaire, 1re et 2e années). Paris, Nathan, 1906. 16mo. 192 pp. F. 0.75

NEUMANN (A.). Resultate zur Aufgabensammlung in Mocniks Lehrbuch der Arithmetik und Algebra für die oberen Klassen der Gymnasien. Wien, Tempsky, 1906. 8vo. 85 pp. Boards. M. 1.20

PEDOTH (A.). Entstehung des Begriffes der ganzen und der gebrochenen Zahlen und die einfachen Rechnungsarten mit diesen. (Progr.) Reichenberg, 1906. 8vo. 22 pp.

ROBBINS (E.). Plane geometry. New York, American Book Company, 1906. 12mo. 254 pp. Cloth. $0.75

ROSS (P.). Elementary algebra. For use of higher grade and secondary schools. Part 1. With answers. London, Longmans, 1906. 8vo. 324 pp. Cloth. 2s. 6d.

SACHSE (J. J.). Zur mechanischen Drittelung eines Winkels und die planimetrische Bestimmung eines Grades der Kreislinie. Heiligenstadt, Cordier, 1906. 8vo. 39 pp. 2 plates. M. 1.20

SOLUTIONS des exercises et problèmes proposés dans le cours moyen de géométrie. 2e édition. Paris, Vitte, 1906. 16mo. 251 pp.

STONE (J. C.) and MILLIS (J. F.). A first algebra for beginners. Boston, Sanborn, 1906. 12mo. 4 + 173 pp. Cloth. $0.50

TAYLOR (J. M.). Plane and spherical trigonometry. Teachers' edition. Boston, Ginn, 1906. 12mo. 3 + 209 pp. Cloth. $1.00

TUÑÓN DE LARA (M.). Lecciones de geometria. Toledo, Serrano, 1906. 8 + 282 pp.

WALLENTIN (F.). Methodisch geordnete Sammlung von Beispielen und Aufgaben aus der Algebra und allgemeinen Arithmetik für die oberen Klassen der Mittelschulen, Lehrer- Bildungsanstalten und anderen gleichstehenden Lehranstalten. Wien, Gerold, 1906. 8vo. 4 + 292 pp. Cloth. M. 3.80

WILLIAMSON (A. P. W.). Plane and spherical trigonometry. With applications to problems in navigation and nautical astronomy. London, Imray, 1906. 8vo. 72 pp. 3s. 6d.

WORKMAN (W. P.) and CRACKNELL (A. G.). Geometry, theoretical and practical. Section II: Rectilinear figures. Section III: The circle.

Section IV: Rectangles and polygons. London, Clive, 1906. 8vo. Cloth. (University tutorial series.) 3s. 6d.

III. APPLIED MATHEMATICS.

BARMWATER (F.). Laerebog i mekanisk fysik. 2te udgave. Kjöbenhaven, 1906. 8vo. 196 pp. M. 5.00

BÖHMER (P. E.) und GRAMBERG (W.). Der Risikogewinn in der Lebens- und in der Invalidítäts-Versicherung. Berlin, 1906. 8vo. 43 pp. M. 2.00

CLAISE (B.) et DUBOIS (A.). Cours théorique et pratique de résistance des matériaux. Bruxelles, 1906. 8vo. 199 pp. Fr. 5.00

CROIX (L.). La règle à calculs appliquée aux problèmes de la navigation. Paris, 1906. 8vo. Fr. 1.50

DORRIE (H.). Das Gesetz von Biot und Savart und die cyclische Konstante. Biedenkopf, 1906. 8vo. 20 pp. M. 1.50

DUBOIS (A.). See CLAISE (B.).

FRANKLIN (W. S.) and ESTY (W.). The elements of electrical engineering. A text-book for schools and colleges. Vol. I. Direct current machines, electric distribution and lighting. New York, Macmillan, 1906. 8vo. 13 + 517 pp. Cloth. $4.50

GRAMBERG (W.). See BÖHMER (P. E.).

HORART (H. M.). Elementary principles of continuous current dynamo design. London, Whittaker, 1906. 8vo. 730 pp. 7s. 6d.

JADANZA (N.). Teorica dei canocchiali esposta secondo il metodo di Gauss. 2a edizione. Torino, Loescher, 1906. 8vo. 11 + 282 pp.

JAMIESON (A.). Text-book on applied mechanics and mechanical engineering. Vol. 2. 5th edition, revised. London, Griffin, 1906. 8vo. 834 pp. 12s. 6d.

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